

Symbolic computation of weighted Moore-Penrose inverse using partitioning method

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Abstract

We propose a method and algorithm for computing the weighted Moore-Penrose inverse of one-variable rational matrices. Continuing this idea, we develop an algorithm for computing the weighted Moore-Penrose inverse of one-variable polynomial matrix. These methods and algorithms are generalizations of the method of computing the weighted Moore-Penrose inverse for constant matrices, originated in [28], and the partitioning method for computing the Moore-Penrose inverse of rational and polynomial matrices introduced in [23]. Algorithms are implemented in the symbolic computational package MATHEMATICA.

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1 Introduction

Let \mathbf{C} be the set of complex numbers, $\mathbf{C}^{m \times n}$ be the set of $m \times n$ complex matrices, and $\mathbf{C}_r^{m \times n} = \{X \in \mathbf{C}^{m \times n} : \text{rank}(X) = r\}$. For any matrix $A \in \mathbf{C}^{m \times n}$ and positive definite matrices M and N of the orders m and n respectively, consider the following equations in X , where $*$ denotes conjugate and transpose:

$$\begin{aligned} (1) \quad & AXA = A & (2) \quad & XAX = X \\ (3M) \quad & (MAX)^* = MAX & (4N) \quad & (NXA)^* = NXA. \end{aligned}$$

The matrix X satisfying (1), (2), (3M) and (4N) is called the weighted Moore-Penrose inverse of A , and it is denoted by $X = A_{MN}^\dagger$. Especially, in the case

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$M = I_m$ and $N = I_n$, the matrix $X = A_{MN}^\dagger$ comes to the Moore-Penrose inverse of A , and it is denoted by $X = A^\dagger$.

For any matrix $A \in \mathbf{C}^{n \times n}$ the Drazin inverse of A is the unique matrix, denoted by A^D , and satisfying the matrix equation (2) and the following equation in X :

$$(1^k) \quad A^k X A = A^k, \quad (5) \quad AX = XA.$$

As usual, $\mathbf{C}[s]$ (resp. $\mathbf{C}(s)$) denotes the polynomials (resp. rational functions) with complex coefficients in the indeterminate s . The $m \times n$ matrices with elements in $\mathbf{C}[s]$ (resp. $\mathbf{C}(s)$) are denoted by $\mathbf{C}[s]^{m \times n}$ (resp. $\mathbf{C}(s)^{m \times n}$). By I is denoted an appropriate identity matrix.

We observed three different directions in the symbolic computation of generalized inverses:

- A) extensions of Leverrier-Faddeev algorithm,
- B) methods based on the interpolation, and
- C) methods based on the Greville's recursive algorithm.

A) Computation of the Moore-Penrose inverse of one variable polynomial and/or rational matrices, based on the Leverrier-Faddeev algorithm, is investigated in [1, 9, 11, 12, 18, 25]. These papers are based on the paper [5]. Implementation of the algorithm from [12] in the symbolic computational language MAPLE, is described in [11]. An algorithm for computing the Moore-Penrose inverse of two-variable rational and polynomial matrix is introduced in [16]. A quicker and less memory-expensive effective algorithm for computing the Moore-Penrose inverse of one-variable and two-variable polynomial matrix, with respect to those introduced in [12] and [16], is presented in [14]. This algorithm is efficient when elements of the input matrix are polynomials with only few nonzero addends.

Continuing the algorithm of the Leverrier-Faddeev type for computing the Drazin inverse of constant matrices, established in [6], a representation and corresponding algorithm for computing the Drazin inverse of a nonregular polynomial matrix of an arbitrary degree is introduced in [10], [22, 25]. Bu and Wei in [3] proposed a finite algorithm for symbolic computation of the Drazin inverse of two-variable rational and polynomial matrices. Also, a more effective three-dimensional version of these algorithms is presented in the paper [3]. Implementation of this algorithm in the programming language MATLAB is presented in [3].

A general for of the Leverrier-Faddeev type algorithms is introduced in [24]. This algorithm generates the class of outer inverses of a rational or polynomial matrix.

B) An interpolation algorithm for computing the Moore-Penrose inverse of a given one-variable polynomial matrix, based on the Leverrier-Faddeev method, is presented in [20]. Algorithms for computing the Moore-Penrose and the Drazin inverse of one-variable polynomial matrices based on the evaluation-interpolation technique and the Fast Fourier transform are introduced in [15].

Corresponding algorithms for two-variable polynomial matrices are introduced in [27]. These algorithms are efficient when the input matrix is dense.

C) Greville's *partitioning method* for numerical computation of generalized inverses is introduced in [7]. Two different proofs for Greville's method were presented in [4], [29]. A simple derivation of the Greville's result has been given by Udawadia and Kalaba [26]. In [8] Fan and Kalaba used the approach of determination of the Moore-Penrose inverse of matrices using dynamic programming and Belman's principle of optimality. Wang in [28] generalizes Greville's method to the weighted Moore-Penrose inverse. Also, the results in [28] are proved using a new technique.

In [21] the Greville's algorithm is estimated as the method which needs more operations and consequently it accumulates more rounding errors. Moreover, it is well-known that the Moore-Penrose inverse is not necessarily a continuous function of the elements of the matrix. The existence of this discontinuity present further problems in the pseudoinverse computation. It is therefore clear that cumulative round off errors should be totally eliminated. During the symbolic implementation, variables are stored in the "exact" form or can be left "unassigned" (without numerical values), resulting in no loss of accuracy during the calculation [12].

An algorithm for computing the Moore-Penrose inverse of one-variable polynomial and/or rational matrices, based on the Greville's partitioning algorithm, was introduced in [23]. An extension of results from [23] to the set of two-variable rational and polynomial matrices is introduced in the paper [19].

In the present paper we extend Wang's partition method from [28] to the set of one-variable rational and polynomial matrices. In this way, we obtain an algorithm for computing the weighted Moore-Penrose inverse of one-variable rational and polynomial matrices. The paper is a generalization of the paper [28] and a continuation of the paper [23].

The structure of the paper is as follows. In the second section we extend the algorithm for computing the weighted Moore-Penrose from [28] to the set of one-variable rational matrices. In Section 3 we give the main theorem and adapt this algorithm to the set of polynomial matrices. Several symbolic examples are arranged in fourth section. In partial case $M = I_m$, $N = I_n$ we obtain the usual Moore-Penrose inverse, and then use test examples from [32]. In the last section we describe main implementation details.

2 Weighted Moore-Penrose inverse for rational matrices

Greville in [7] proposed the partitioning algorithm which relates the Moore-Penrose pseudoinverse of a constant matrix R augmented by a vector r of appropriate dimensions with the pseudoinverse R^\dagger of R . Wang and Chen in [28] generalize Greville's partitioning method. They obtained an algorithm for com-

puting the weighted Moore-Penrose inverse, and give a new technique for its proof. This method is also suitable for the weighted least-squares problem.

By $\widehat{A}_i(s)$ we denote the submatrix of $A(s) \in \mathbf{C}(s)^{m \times n}$ consisting of its first i columns:

$$\widehat{A}_i(s) = \left[\widehat{A}_{i-1}(s) \mid a_i(s) \right], i = 2, \dots, n, \quad \widehat{A}_1(s) = a_1(s) \quad (2.1)$$

where $a_i(s)$ is the i -th column of A .

In the sequel we consider positive definite matrices $M(s) \in \mathbf{C}(s)^{m \times m}$ and $N(s) \in \mathbf{C}(s)^{n \times n}$. The leading principal submatrix $N_i(s) \in \mathbf{C}(s)^{i \times i}$ of $N(s)$ is partitioned as

$$N_i(s) = \begin{bmatrix} N_{i-1}(s) & l_i(s) \\ l_i^*(s) & n_{ii}(s) \end{bmatrix}, \quad i = 2, \dots, n. \quad (2.2)$$

In the following lemma we generalize the representation of the weighted Moore-Penrose inverse from [28] to the set of one-variable rational matrices.

For the sake of simplicity, by $X_i(s)$ we denote the weighted Moore-Penrose inverse corresponding to submatrices $\widehat{A}_i(s)$ and $N_i(s)$: $X_i(s) = \widehat{A}_i(s)^\dagger_{MN_i}$, for each $i = 1, \dots, n$.

Lemma 2.1. *Let $A(s) \in \mathbf{C}(s)^{m \times n}$, assume that $M(s) \in \mathbf{C}(s)^{m \times m}$ and $N(s) \in \mathbf{C}(s)^{n \times n}$ are positive definite matrices, and let $\widehat{A}_i(s)$ be the submatrix of $A(s)$ consisting of its first i columns. Assume that the leading principal submatrix of $N(s)$, denoted by $N_i(s) \in \mathbf{C}(s)^{i \times i}$, is partitioned as in (2.2).*

In the case $i = 1$ we have

$$X_1(s) = a_1(s)^\dagger = \begin{cases} (a_1^*(s)M(s)a_1(s))^{-1} a_1^*(s)M(s), & a_1(s) \neq 0, \\ a_1^*(s), & a_1(s) = 0. \end{cases} \quad (2.3)$$

For each $i = 2, \dots, n$ $X_i(s)$ is equal to

$$X_i(s) = \begin{bmatrix} X_{i-1}(s) - \left(d_i(s) + (I - X_{i-1}(s)\widehat{A}_{i-1}(s)) N_{i-1}^{-1}(s) l_i(s) \right) b_i^*(s) \\ b_i^*(s) \end{bmatrix} \quad (2.4)$$

where the vectors $d_i(s)$, $c_i(s)$ and $b_i^(s)$ are defined by*

$$d_i(s) = X_{i-1}(s)a_i(s) \quad (2.5)$$

$$c_i(s) = a_i(s) - \widehat{A}_{i-1}(s)d_i(s) = \left(I - \widehat{A}_{i-1}(s)X_{i-1}(s) \right) a_i(s). \quad (2.6)$$

$$b_i^*(s) = \begin{cases} (c_i^*(s)M(s)c_i(s))^{-1} c_i^*(s)M(s), & c_i(s) \neq 0 \\ \delta_i^{-1}(s) (d_i^*(s)N_{i-1}(s) - l_i(s)^*) X_{i-1}(s), & c_i(s) = 0, \end{cases} \quad (2.7)$$

and where in (2.7) is

$$\begin{aligned} \delta_i(s) &= n_{ii}(s) + d_i^*(s)N_{i-1}(s)d_i(s) - (d_i^*(s)l_i(s) + l_i^*(s)d_i(s)) \\ &\quad - l_i^*(s) \left(I - X_{i-1}(s)\widehat{A}_{i-1}(s) \right) N_{i-1}^{-1}(s) l_i(s). \end{aligned} \quad (2.8)$$

Proof. The proof is the same as for constant matrices, presented in [28]. \square

The following lemma is a simple extension of the well-known result in the literature.

Lemma 2.2. *Let $A(s)$ be a partitioned matrix which is nonsingular, and let the submatrix $A_{11}(s)$ also be nonsingular. Then*

$$A(s) = \begin{bmatrix} A_{11}(s) & A_{12}(s) \\ A_{21}(s) & A_{22}(s) \end{bmatrix}^{-1} = \begin{bmatrix} B_{11}(s) & B_{12}(s) \\ B_{21}(s) & B_{22}(s) \end{bmatrix} \quad (2.9)$$

where

$$B_{11}(s) = A_{11}(s)^{-1} + A_{11}(s)^{-1}A_{12}(s)B_{22}(s)A_{21}(s)A_{11}(s)^{-1} \quad (2.10)$$

$$B_{12}(s) = -A_{11}(s)^{-1}A_{12}(s)B_{22}(s) \quad (2.11)$$

$$B_{21}(s) = -B_{22}(s)A_{21}(s)A_{11}(s)^{-1} \quad (2.12)$$

$$B_{22}(s) = (A_{22}(s) - A_{21}A_{11}(s)^{-1}A_{12}(s))^{-1}. \quad (2.13)$$

The following lemma is a generalization of known result from [28] to the set of rational matrices.

Lemma 2.3. *Let $N_i(s)$ be the partitioned matrix defined in (2.2). Assume that $N_i(s)$ and $N_{i-1}(s)$ are both nonsingular. Then*

$$N_i^{-1}(s) = \begin{bmatrix} N_{i-1}(s) & l_i(s) \\ l_i^*(s) & n_{ii}(s) \end{bmatrix}^{-1} = \begin{bmatrix} E_{i-1}(s) & f_i(s) \\ f_i^*(s) & g_{ii}(s) \end{bmatrix} \quad (2.14)$$

where

$$g_{ii}(s) = (n_{ii}(s) - l_i^*(s)N_{i-1}^{-1}(s)l_i(s))^{-1} \quad (2.15)$$

$$f_i(s) = -g_{ii}(s)N_{i-1}^{-1}(s)l_i(s) \quad (2.16)$$

$$E_{i-1}(s) = N_{i-1}^{-1}(s) + g_{ii}^{-1}(s)f_i(s)f_i^*(s). \quad (2.17)$$

Proof. The proof immediately follows from the substitutions $A_{11}(s) = N_{i-1}(s)$, $A_{12}(s) = l_i(s)$, $A_{21}(s) = l_i^*(s)$ and $A_{22}(s) = n_{ii}(s)$ in Lema 2.2. \square

In view of Lemma 2.1 we present the following algorithm for computing the weighted Moore-Penrose inverse of a given one-variable rational matrix.

Algorithm 2.1. *Input: rational matrix $A(s) \in \mathbf{C}(s)^{m \times n}$ and positive definite matrices $M(s) \in \mathbf{C}(s)^{m \times m}$ and $N(s) \in \mathbf{C}(s)^{n \times n}$.*

Step 1. Initial value: Compute $X_1(s) = a_1(s)^\dagger$ defined in (2.3).

Step 2. Recursive step: For each $i = 2, \dots, n$ compute $X_i(s)$ performing the following four steps:

Step 2.1. Compute $d_i(s)$ using (2.5).

Step 2.2. Compute $c_i(s)$ using (2.6).

Step 2.3. Compute $b_i^(s)$ by means of (2.7) and (2.8).*

Step 2.4. Applying (2.4) compute $X_i(s)$.

Step 3. The stopping criterion: $i = n$. Return $X_n(s)$.

Let $N_i(s)$, defined in (2.2), be the leading principal submatrix of positive definite matrix $N(s) \in \mathbf{C}(s)^{n \times n}$. The following algorithm, based on Lemma 2.3, computes the inverse matrix $N_i^{-1}(s) \in \mathbf{C}(s)^{i \times i}$.

Algorithm 2.2. *Compute $N_i^{-1}(s)$.*

Step 1. Initial values: $N_1^{-1}(s) = n_{11}^{-1}(s)$

Step 2. Recursive step: For $i = 2, \dots, n$ perform the following steps:

Step 2.1. Compute $g_{ii}(s)$ using (2.15).

Step 2.2. Compute $f_i(s)$ applying (2.16).

Step 2.3. Compute $E_{i-1}(s)$ according to (2.17).

Step 2.4. Compute $N_i^{-1}(s)$ using (2.14).

Step 3. Stopping criterion: for $i = n$ the output is the inverse matrix $N^{-1}(s) = N_n^{-1}(s)$.

3 Weighted Moore-Penrose inverse for polynomial matrices

Consider the matrix $A(s) \in \mathbf{C}[s]^{m \times n}$ given in the polynomial form with respect to unknown s :

$$A(s) = A_1 + A_2s + \dots + A_qs^{q-1} + A_{q+1}s^q = \sum_{i=0}^q A_{i+1}s^i \quad (3.1)$$

where A_i , $i = 1, \dots, q+1$ are constant $m \times n$ matrices.

Theorem 3.1. *Consider an arbitrary polynomial matrix $A(s) \in \mathbf{C}[s]^{m \times n}$ given by (3.1) and the following polynomial forms of positive definite matrices $M(s) \in \mathbf{C}[s]^{m \times m}$ and $N(s) \in \mathbf{C}[s]^{n \times n}$:*

$$M(s) = \sum_{i=0}^{m_q} M_{i+1}s^i, \quad N(s) = \sum_{i=0}^{n_q} N_{i+1}s^i. \quad (3.2)$$

Transcribe i -th column of $A(s)$ by

$$a_i(s) = \sum_{j=0}^q a_{i,j+1}s^j, \quad 1 \leq i \leq n, \quad (3.3)$$

where $a_{i,j+1}$, $0 \leq j \leq q$ are constant $m \times 1$ vectors. Also, denote first i columns of $A(s)$ by

$$\hat{A}_i(s) = \sum_{j=0}^q \hat{A}_{i,j+1}s^j, \quad 1 \leq i \leq n, \quad (3.4)$$

where $\hat{A}_{i,j+1}$, $0 \leq j \leq q$ are constant $m \times i$ matrices.

In the partition (2.2) of the leading principal submatrix $N_i(s) \in \mathbb{C}[s]^{i \times i}$ of $N(s)$, we use the following polynomial representations:

$$n_{ii}(s) = \sum_{j=0}^{n_q} \widehat{n}_{i,j+1} s^j, \quad l_i(s) = \sum_{j=0}^{n_q} L_{i,j+1} s^j, \quad N_{i-1}^{-1}(s) = \frac{\sum_{j=0}^{\overline{n}_q} \overline{N}_{i-1,j+1} s^j}{\sum_{j=0}^{\overline{n}_q} \overline{\overline{N}}_{i-1,j+1} s^j}. \quad (3.5)$$

Then the following algorithm computes the weighted Moore-Penrose inverse $A(s)_{M,N}^\dagger$.

Algorithm 3.1.

Step 1. Initial values:

Compute $Z_{1,j+1}$, $0 \leq j \leq q_1 = q + m_q$ and $Y_{1,j+1}$, $0 \leq j \leq p_1 = 2q + m_q$ as in

$$Z_{1,j+1} = \begin{cases} \sum_{k=0}^j a_{1,j-k+1}^* M_{k+1}, & 0 \leq j \leq q + m_q, \quad a_1(s) \neq 0, \\ a_{1,j+1}^* = 0, & a_1(s) = 0. \end{cases} \quad (3.6)$$

$$Y_{1,j+1} = \begin{cases} \sum_{r=0}^j \sum_{k=0}^{j-r} a_{1,j-k-r+1}^* M_{k+1} a_{1,r+1}, & 0 \leq j \leq 2q + m_q, \quad a_1(s) \neq 0, \\ 1, & a_1(s) = 0. \end{cases} \quad (3.7)$$

Step 2. Recursive step:

For $2 \leq i \leq n$ perform Step 2.1, Step 2.2, Step 2.3 and Step 2.4.

Step 2.1. Compute $d_{i,j+1}$, $0 \leq j \leq q + q_{i-1}$ by means of

$$d_{i,j+1} = \sum_{k=0}^j Z_{i-1,j-k+1} a_{i,k+1}, \quad 0 \leq j \leq q_{i-1} + q. \quad (3.8)$$

Step 2.2. Compute $c_{i,j+1}$, $0 \leq j \leq \hat{q}_{i-1} + q$ using

$$c_{i,j+1} = \sum_{k=0}^j (a_{i,j-k+1} Y_{i-1,k+1} - \widehat{A}_{i-1,j-k+1} d_{i,k+1}), \quad 0 \leq j \leq \hat{q}_{i-1} + q. \quad (3.9)$$

where

$$\hat{q}_{i-1} = \max\{p_{i-1}, q_{i-1} + q\}. \quad (3.10)$$

Step 2.3. If $c_{i,j+1} \neq 0$ for some j , compute $V_{i,j+1}$ and $W_{i,j+1}$ by means of

$$V_{i,j+1} = \sum_{r=0}^j \sum_{k=0}^{j-r} Y_{i-1,j-k-r+1} c_{i,k+1}^* M_{r+1}, \quad (3.11)$$

$$0 \leq j \leq \bar{b}_i = \hat{q}_{i-1} + q + p_{i-1} + m_q,$$

$$W_{i,j+1} = \sum_{r=0}^j \sum_{k=0}^{j-r} c_{i,j-k-r+1}^* M_{k+1} c_{i,r+1}, \quad (3.12)$$

$$0 \leq j \leq \bar{\bar{b}}_i = 2\hat{q}_{i-1} + 2q + m_q.$$

In the case $c_{i,j+1} = 0$ for each j , compute $V_{i,j+1}$ and $W_{i,j+1}$ in this way:

$$V_{i,j+1} = \sum_{t=0}^j \sum_{r=0}^{j-t} \sum_{k=0}^{j-t-r} \bar{\Delta}_{i,j-k-r-t+1} d_{i,t+1} N_{i-1,r+1} Z_{i-1,k+1} - \bar{\Delta}_{i,j-k-r-t+1} L_{i,t+1}^* Y_{i-1,r+1} Z_{i-1,k+1}, \quad (3.13)$$

$$0 \leq j \leq \bar{b}_i = 2p_{i-1} + \bar{n}_q + q_{i-1} + \hat{q}_{i-1} + n_q,$$

$$W_{i,j+1} = \sum_{r=0}^j \sum_{k=0}^{j-r} \bar{\Delta}_{i,j-k-r+1} Y_{i-1,k+1} Y_{i-1,r+1}, \quad (3.14)$$

$$0 \leq j \leq \bar{\bar{b}}_i = 2\hat{q}_{i-1} + n_q + \max\{n_q + \bar{n}_q, \bar{\bar{n}}_q\} + 2p_{i-1},$$

where

$$\bar{\Delta}_{i,j+1} = \sum_{r=0}^j \sum_{k=0}^{j-r} Y_{i-1,j-k-r+1} Y_{i-1,k+1} \bar{\bar{N}}_{i-1,r+1} s^j, \quad 0 \leq j \leq \bar{\delta}_q = 2p_{i-1} + \bar{n}_q \quad (3.15)$$

$$\begin{aligned} \bar{\bar{\Delta}}_{i,j+1} = & \sum_{t=0}^j \sum_{r=0}^{j-t} \sum_{k=0}^{j-r-t} (\hat{n}_{i,j-k-r-t+1} Y_{i-1,k+1} Y_{i-1,r+1} + d_{i,j-k-r-t+1}^* N_{i-1,k+1} d_{i,r+1} \\ & - d_{i,j-k-r-t+1}^* L_{i,k+1} Y_{i-1,r+1} - L_{i,j-k-r-t+1}^* d_{i,k+1} Y_{i-1,r+1}) \bar{\bar{N}}_{i-1,t+1} \\ & - L_{i,j-k-r+1}^* \varphi_{i,k+1} Y_{i-1,r+1}, \quad 0 \leq j \leq \bar{\bar{\delta}}_q = 2\hat{q}_{i-1} + n_q + \max\{n_q + \bar{n}_q, \bar{\bar{n}}_q\} \end{aligned} \quad (3.16)$$

Then compute

$$Z_{i,j+1}, \quad 0 \leq j \leq q_i, \quad Y_{i,j+1}, \quad 0 \leq j \leq p_i$$

as it is defined in

$$Z_{i,j+1} = \begin{bmatrix} \Theta_{i,j+1} \\ \sum_{k=0}^j \psi_{i,j-k+1} V_{i,k+1} \end{bmatrix}, \quad 0 \leq j \leq q_i, \quad i \geq 2 \quad (3.17)$$

$$Y_{i,j+1} = \sum_{k=0}^j \psi_{i,j-k+1} W_{i,k+1}, \quad 0 \leq j \leq p_i, \quad i \geq 2, \quad (3.18)$$

where in (3.17) and (3.18) is:

$$\begin{aligned}\Theta_{i,j+1} = & \sum_{r=0}^j \sum_{k=0}^{j-r} Z_{i-1,j-k-r+1} \overline{\overline{N}}_{i-1,k+1} W_{i,r+1} \\ & - d_{i,j-k-r+1} \overline{\overline{N}}_{i-1,k+1} V_{i,r+1} - \varphi_{i,j-k+1} V_{i,k+1}, \\ & 0 \leq j \leq q_i,\end{aligned}\quad (3.19)$$

$$\begin{aligned}\varphi_{i,j+1} = & \sum_{t=0}^j \sum_{r=0}^{j-t} \sum_{k=0}^{j-t-r} \left(Y_{i-1,j-k-r-t+1} I_{i-1,k+1} - Z_{i-1,j-k-r-t+1} \hat{A}_{i-1,k+1} \right) \\ & \times \overline{\overline{N}}_{i-1,r+1} L_{i,k+1}, \\ & 0 \leq j \leq \hat{q}_{i-1} + \overline{n}_q + n_q,\end{aligned}\quad (3.20)$$

$$\psi_{i,j+1} = \sum_{k=0}^j Y_{i-1,j-k+1} \overline{\overline{N}}_{i-1,k+1}, \quad 0 \leq j \leq p_{i-1} + \overline{n}_q \quad (3.21)$$

and

$$q_i = \hat{q}_{i-1} + q + \max\{\overline{n}_q + n_q, \overline{\overline{n}}_q\} + \max\{\overline{b}_i, \overline{\overline{b}}_i\}, \quad q_1 = q + m_q \quad (3.22)$$

$$p_i = p_{i-1} + \overline{\overline{n}}_q + \overline{\overline{b}}_i, \quad p_1 = 2q + m_q. \quad (3.23)$$

Step 2.4. Compute

$$X_i(s) = \frac{\sum_{j=0}^{q_i} Z_{i,j+1} s^j}{\sum_{j=0}^{p_i} Y_{i,j+1} s^j}, \quad (3.24)$$

where q_i and p_i are defined in (3.22) and (3.23), respectively.

Step 3. The stopping criterion is $i = n$. In this case the result is the weighted Moore-Penrose inverse $A(s)_{M,N}^\dagger = X_n(s)$.

Proof. If $a_1(s) = 0$, in view of the second case in (2.3) we have

$$X_1(s) = a_1(s)^\dagger = \sum_{j=0}^q a_{1,j+1}^*, \quad Y_1(s) = 1.$$

If $a_1(s) \neq 0$, in accordance with the first case in (2.3) we have

$$\begin{aligned}
X_1(s) &= (a_1^*(s)M(s)a_1(s))^{-1} a_1^*(s)M(s) \\
&= \left(\sum_{j=0}^q a_{1,j+1}^* s^j \sum_{j=0}^{m_q} M_{j+1} s^j \sum_{j=0}^q a_{1,j+1} s^j \right)^{-1} \sum_{j=0}^q a_{1,j+1}^* s^j \sum_{j=0}^{m_q} M_{j+1} s^j \\
&= \left(\sum_{j=0}^{2q+m_q} \sum_{r=0}^j \sum_{k=0}^{j-r} a_{1,j-k-r+1}^* M_{k+1} a_{1,r+1} s^j \right)^{-1} \sum_{j=0}^{q+m_q} \sum_{k=0}^j a_{1,j-k+1}^* M_{k+1} s^j \\
&= \frac{\sum_{j=0}^{q+m_q} \sum_{k=0}^j a_{1,j-k+1}^* M_{k+1} s^j}{\sum_{j=0}^{2q+m_q} \sum_{r=0}^j \sum_{k=0}^{j-r} a_{1,j-k-r+1}^* M_{k+1} a_{1,r+1} s^j}.
\end{aligned}$$

Therefore, $X_1(s)$ is the partial case $i = 1$ of (3.24), where the matrices $Z_{1,j+1}$ and $Y_{1,j+1}$ are defined in (3.6) and (3.7), respectively.

For each $i = 2, \dots, n$ it is reasonable to calculate matrices $X_i(s)$ in the form (3.24), for appropriate matrices $Z_{i,j+1}$, $Y_{i,j+1}$ and appropriate upper bounds q_i and p_i .

Direct calculation in (2.5), i.e. *Step 2.1* of *Algorithm 2.1* yields the following

$$\begin{aligned}
d_i(s) &= X_{i-1}(s)a_i(s) = \frac{\sum_{j=0}^{q_{i-1}} Z_{i-1,j+1} s^j}{\sum_{j=0}^{p_{i-1}} Y_{i-1,j+1} s^j} \cdot \sum_{k=0}^q a_{i,k+1} s^k \\
&= \frac{\sum_{j=0}^{q_{i-1}+q} \left(\sum_{k=0}^j Z_{i-1,j-k+1} a_{i,k+1} \right) s^j}{\sum_{j=0}^{p_{i-1}} Y_{i-1,j+1} s^j}.
\end{aligned}$$

Then $d_i(s)$ can be represented in the form

$$d_i(s) = \frac{\sum_{j=0}^{q_{i-1}+q} d_{i,j+1} s^j}{\sum_{j=0}^{p_{i-1}} Y_{i-1,j+1} s^j}, \tag{3.25}$$

where the matrices $d_{i,j+1}$ are defined by (3.8).

Consider (2.6), i.e. *Step 2.2* of *Algorithm 2.1*. Since the first $i - 1$ columns of $A(s)$ can be represented in the polynomial form

$$\hat{A}_{i-1}(s) = \sum_{j=0}^q \hat{A}_{i-1,j+1} s^j$$

for appropriate $m \times (i-1)$ constant matrices $\widehat{A}_{i-1,j+1}(s)$, in view of (3.3) and (3.25) we obtain

$$\begin{aligned}
 c_i(s) &= a_i(s) - \widehat{A}_{i-1}(s)d_i(s) \\
 &= \sum_{j=0}^q a_{i,j+1}s^j - \sum_{j=0}^q \widehat{A}_{i-1,j+1}s^j \cdot \frac{\sum_{j=0}^{q_{i-1}+q} d_{i,j+1}s^j}{\sum_{j=0}^{p_{i-1}} Y_{i-1,j+1}s^j} \\
 &= \frac{\sum_{j=0}^{q+p_{i-1}} \left(\sum_{k=0}^j a_{i,j-k+1} Y_{i-1,k+1} \right) s^j - \sum_{j=0}^{2q+q_{i-1}} \left(\sum_{k=0}^j \widehat{A}_{i-1,j-k+1} d_{i,k+1} \right) s^j}{\sum_{j=0}^{p_{i-1}} Y_{i-1,j+1}s^j}
 \end{aligned}$$

Finding a maximum between the upper bounds $q + p_{i-1}$ and $2q + q_{i-1}$ in the last identity, we have

$$c_i(s) = \frac{\sum_{j=0}^{\hat{q}_{i-1}+q} \sum_{k=0}^j (a_{i,j-k+1} Y_{i-1,k+1} - \widehat{A}_{i-1,j-k+1} d_{i,k+1}) s^j}{\sum_{j=0}^{p_{i-1}} Y_{i-1,j+1}s^j}, \quad ,$$

where \hat{q}_{i-1} is defined in (3.10) and shorter polynomial matrix is filled by appropriate zero matrices.

Therefore, $c_i(s)$ can be represented in the form

$$c_i(s) = \frac{\sum_{j=0}^{\hat{q}_{i-1}+q} c_{i,j+1}s^j}{\sum_{j=0}^{p_{i-1}} Y_{i-1,j+1}s^j},$$

where $c_{i,j+1}$ are matrices of the form (3.9), for each $0 \leq j \leq \hat{q}_{i-1} + q$.

Observe now *Step 2.3.* of *Algorithm 2.1*, i.e (2.6).

If $c_{i,j+1} \neq 0$ for some j , then $c_i(s) \neq 0$ and $b_i^*(s)$ is equal to

$$\begin{aligned}
b_i^*(s) &= (c_i^*(s)M(s)c_i(s))^{-1} c_i^*(s)M(s) \\
&= \left[\frac{\sum_{j=0}^{\hat{q}_{i-1}+q} c_{i,j+1}^* s^j}{\sum_{j=0}^{p_{i-1}} Y_{i-1,j+1} s^j} \sum_{j=0}^{m_q} M_{j+1} s^j \frac{\sum_{j=0}^{\hat{q}_{i-1}+q} c_{i,j+1} s^j}{\sum_{j=0}^{p_{i-1}} Y_{i-1,j+1} s^j} \right]^{-1} \frac{\sum_{j=0}^{\hat{q}_{i-1}+q} c_{i,j+1}^* s^j}{\sum_{j=0}^{p_{i-1}} Y_{i-1,j+1} s^j} \sum_{j=0}^{m_q} M_{j+1} s^j \\
&= \frac{\sum_{j=0}^{p_{i-1}} Y_{i-1,j+1} s^j \cdot \sum_{j=0}^{\hat{q}_{i-1}+q+m_q} \sum_{k=0}^j c_{i,j-k+1}^* M_{k+1} s^j}{\sum_{j=0}^{2\hat{q}_{i-1}+2q+m_q} \sum_{r=0}^j \sum_{k=0}^{j-r} c_{i,j-k-r+1}^* M_{k+1} c_{i,r+1} s^j} \\
&= \frac{\sum_{j=0}^{\hat{q}_{i-1}+q+p_{i-1}+m_q} \sum_{r=0}^j \sum_{k=0}^{j-r} Y_{i-1,j-k-r+1} c_{i,k+1}^* M_{r+1} s^j}{\sum_{j=0}^{2\hat{q}_{i-1}+2q+m_q} \sum_{r=0}^j \sum_{k=0}^{j-r} c_{i,j-k-r+1}^* M_{k+1} c_{i,r+1} s^j} = \frac{\sum_{j=0}^{\bar{b}_i} V_{i,j+1} s^j}{\sum_{j=0}^{\bar{b}_i} W_{i,j+1} s^j}
\end{aligned}$$

where $V_{i,j+1}$ and $W_{i,j+1}$ satisfy (3.11) and (3.12), respectively.

If $c_{i,j+1} = 0$ for all j , then $c_i(s) = 0$ and $b_i^*(s)$ is defined in the second case of (2.7) and in (2.8). In order to compute $\delta_i^{-1}(s)$, we firstly generate the following intermediate value, which will be used later:

$$\begin{aligned}
\sigma_i(s) &= (I - X_{i-1}(s)\hat{A}_{i-1}(s)) N_{i-1}^{-1}(s) l_i(s) \\
&= \left(I - \frac{\sum_{j=0}^{q_{i-1}} Z_{i-1,j+1} s^j}{\sum_{j=0}^{p_{i-1}} Y_{i-1,j+1} s^j} \sum_{j=0}^q \hat{A}_{i-1,j+1} s^j \right) \frac{\sum_{j=0}^{\bar{n}_q} \bar{N}_{i-1,j+1} s^j}{\sum_{j=0}^{\bar{n}_q} \bar{N}_{i-1,j+1} s^j} \sum_{j=0}^{n_q} L_{i,j+1} s^j \\
&= \frac{\sum_{j=0}^{\hat{q}_{i-1}} \sum_{k=0}^j Y_{i-1,j-k+1} I_{i-1,k+1} - Z_{i-1,j-k+1} \hat{A}_{i-1,k+1} s^j}{\sum_{j=0}^{p_{i-1}} Y_{i-1,j+1} s^j} \cdot \frac{\sum_{j=0}^{\bar{n}_q+n_q} \sum_{k=0}^j \bar{N}_{i-1,j-k+1} L_{i,k+1} s^j}{\sum_{j=0}^{\bar{n}_q} \bar{N}_{i-1,j+1} s^j} \\
&= \frac{\sum_{j=0}^{\hat{q}_{i-1}+\bar{n}_q+n_q} \sum_{t=0}^j \sum_{r=0}^{j-t} \sum_{k=0}^{j-t-r} (Y_{i-1,j-k-r-t+1} I_{i-1,k+1} - Z_{i-1,j-k-r-t+1} \hat{A}_{i-1,k+1}) \bar{N}_{i-1,r+1} L_{i,k+1}}{\sum_{j=0}^{p_{i-1}+\bar{n}_q} \sum_{k=0}^j Y_{i-1,j-k+1} \bar{N}_{i-1,k+1} s^j} \\
&= \frac{\sum_{j=0}^{\hat{q}_{i-1}+\bar{n}_q+n_q} \varphi_{i,j+1} s^j}{\sum_{j=0}^{p_{i-1}+\bar{n}_q} \psi_{i,j+1} s^j}.
\end{aligned}$$

In the last identity $\varphi_{i,j+1}$ and $\psi_{i,j+1}$ and δ_i are defined by (3.20) and (3.21).

Now, $\delta_i(s)$ is equal to

$$\begin{aligned}
\delta_i(s) &= n_{ii}(s) + d_i^*(s)N_{i-1}(s)d_i(s) - (d_i^*(s)l_i(s) + l_i^*(s)d_i(s)) - l_i^*(s)\sigma_i(s) \\
&= \sum_{j=0}^{n_q} \hat{n}_{i,j+1} s^j + \frac{\sum_{j=0}^{q_{i-1}+q} d_{i,j+1}^* s^j}{\sum_{j=0}^{p_{i-1}} Y_{i-1,j+1}} \sum_{j=0}^{n_q} N_{i-1,j+1} s^j \frac{\sum_{j=0}^{q_{i-1}+q} d_{i,j+1} s^j}{\sum_{j=0}^{p_{i-1}} Y_{i-1,j+1}} \\
&\quad - \frac{\sum_{j=0}^{q_{i-1}+q} d_{i,j+1}^* s^j}{\sum_{j=0}^{p_{i-1}} Y_{i-1,j+1}} \sum_{j=0}^{n_q} L_{i,j+1} s^j - \sum_{j=0}^{n_q} L_{i,j+1}^* s^j \frac{\sum_{j=0}^{q_{i-1}+q} d_{i,j+1} s^j}{\sum_{j=0}^{p_{i-1}} Y_{i-1,j+1}} - \sum_{j=0}^{n_q} L_{i,j+1}^* s^j \frac{\sum_{j=0}^{\hat{q}_{i-1}+\overline{n}_q+n_q} \varphi_{i,j+1} s^j}{\sum_{j=0}^{p_{i-1}+\overline{n}_q} \psi_{i,j+1} s^j} \\
&= \frac{\sum_{j=0}^{2\hat{q}_{i-1}+n_q} \sum_{r=0}^j \sum_{k=0}^{j-r} \hat{n}_{i,j-k-r+1} Y_{i-1,k+1} Y_{i-1,r+1} + d_{i,j-k-r+1}^* N_{i-1,k+1} d_{i,r+1} s^j}{\sum_{j=0}^{2p_{i-1}} \sum_{k=0}^j Y_{i-1,j-k+1} Y_{i-1,k+1} s^j} \\
&\quad - \frac{\sum_{j=0}^{q_{i-1}+q+p_{i-1}+n_q} \sum_{r=0}^j \sum_{k=0}^{j-r} d_{i,j-k-r+1}^* L_{i,k+1} Y_{i-1,r+1} + L_{i,j-k-r+1}^* d_{i,k+1} Y_{i-1,r+1} s^j}{\sum_{j=0}^{2p_{i-1}} \sum_{k=0}^j Y_{i-1,j-k+1} Y_{i-1,k+1} s^j} \\
&\quad - \frac{\sum_{j=0}^{\hat{q}_{i-1}+2n_q+\overline{n}_q} \sum_{k=0}^j L_{i,j-k+1}^* \varphi_{i,k+1} s^j}{\sum_{j=0}^{p_{i-1}+\overline{n}_q} \psi_{i,j+1} s^j} \\
&= \frac{\sum_{j=0}^{2\hat{q}_{i-1}+n_q+\overline{n}_q} \sum_{t=0}^j \sum_{r=0}^{j-t} \sum_{k=0}^{j-r-t} (\hat{n}_{i,j-k-r-t+1} Y_{i-1,k+1} Y_{i-1,r+1} + d_{i,j-k-r-t+1}^* N_{i-1,k+1} d_{i,r+1}) \overline{N}_{i-1,t+1} s^j}{\sum_{j=0}^{2p_{i-1}+\overline{n}_q} \sum_{r=0}^j \sum_{k=0}^{j-r} Y_{i-1,j-k-r+1} Y_{i-1,k+1} \overline{N}_{i-1,r+1} s^j} \\
&\quad - \frac{\sum_{j=0}^{2\hat{q}_{i-1}+n_q+\overline{n}_q} \sum_{t=0}^j \sum_{r=0}^{j-t} \sum_{k=0}^{j-r-t} (d_{i,j-k-r-t+1}^* L_{i,k+1} Y_{i-1,r+1} + L_{i,j-k-r-t+1}^* d_{i,k+1} Y_{i-1,r+1}) \overline{N}_{i-1,t+1} s^j}{\sum_{j=0}^{2p_{i-1}+\overline{n}_q} \sum_{r=0}^j \sum_{k=0}^{j-r} Y_{i-1,j-k-r+1} Y_{i-1,k+1} \overline{N}_{i-1,r+1} s^j} \\
&\quad - \frac{\sum_{j=0}^{\hat{q}_{i-1}+2n_q+p_{i-1}+\overline{n}_q} \sum_{r=0}^j \sum_{k=0}^{j-r} L_{i,j-k-r+1}^* \varphi_{i,k+1} Y_{i-1,r+1} s^j}{\sum_{j=0}^{2p_{i-1}+\overline{n}_q} \sum_{r=0}^j \sum_{k=0}^{j-r} Y_{i-1,j-k-r+1} Y_{i-1,k+1} \overline{N}_{i-1,r+1} s^j} \\
&= \frac{\sum_{j=0}^{\overline{\delta}_q} \sum_{t=0}^j \sum_{r=0}^{j-t} \sum_{k=0}^{j-r-t} (\hat{n}_{i,j-k-r-t+1} Y_{i-1,k+1} Y_{i-1,r+1} + d_{i,j-k-r-t+1}^* N_{i-1,k+1} d_{i,r+1}) \overline{N}_{i-1,t+1} s^j}{\sum_{j=0}^{2p_{i-1}+\overline{n}_q} \sum_{r=0}^j \sum_{k=0}^{j-r} Y_{i-1,j-k-r+1} Y_{i-1,k+1} \overline{N}_{i-1,r+1} s^j} \\
&\quad - \frac{\sum_{j=0}^{\overline{\delta}_q} \sum_{t=0}^j \sum_{r=0}^{j-t} \sum_{k=0}^{j-r-t} (d_{i,j-k-r-t+1}^* L_{i,k+1} Y_{i-1,r+1} + L_{i,j-k-r-t+1}^* d_{i,k+1} Y_{i-1,r+1}) \overline{N}_{i-1,t+1} s^j}{\sum_{j=0}^{2p_{i-1}+\overline{n}_q} \sum_{r=0}^j \sum_{k=0}^{j-r} Y_{i-1,j-k-r+1} Y_{i-1,k+1} \overline{N}_{i-1,r+1} s^j} \\
&= \frac{\sum_{j=0}^{\overline{\delta}_q} \sum_{r=0}^j \sum_{k=0}^{j-r} L_{i,j-k-r+1}^* \varphi_{i,k+1} Y_{i-1,r+1} s^j}{\sum_{j=0}^{2p_{i-1}+\overline{n}_q} \sum_{r=0}^j \sum_{k=0}^{j-r} Y_{i-1,j-k-r+1} Y_{i-1,k+1} \overline{N}_{i-1,r+1} s^j} = \frac{\sum_{j=0}^{\overline{\delta}_q} \overline{\Delta}_{i,j+1} s^j}{\sum_{j=0}^{\overline{\delta}_q} \overline{\Delta}_{i,j+1} s^j}.
\end{aligned}$$

Therefore

$$\delta_i(s)^{-1} = \frac{\sum_{j=0}^{\bar{\delta}_q} \bar{\Delta}_{i,j+1} s^j}{\sum_{j=0}^{\bar{\bar{\delta}}_q} \bar{\bar{\Delta}}_{i,j+1} s^j},$$

where $\bar{\Delta}_{i,j+1}$ and $\bar{\bar{\Delta}}_{i,j+1}$ are defined in (3.15) and (3.16), respectively.

Now, in accordance with the second case of (2.7), $b_i(s)$ is equal to

$$\begin{aligned} b_i^*(s) &= \delta_i^{-1}(s) (d_i^*(s) N_{i-1}(s) - l_i^*(s)) X_{i-1}(s) \\ &= \frac{\sum_{j=0}^{\bar{\delta}_q} \bar{\Delta}_{i,j+1} s^j}{\sum_{j=0}^{\bar{\bar{\delta}}_q} \bar{\bar{\Delta}}_{i,j+1} s^j} \left(\frac{\sum_{j=0}^{q_{i-1}+q} d_{i,j+1}^* s^j}{\sum_{j=0}^{p_{i-1}} Y_{i-1,j+1} s^j} \sum_{j=0}^{n_q} N_{i-1,j+1} s^j - \sum_{j=0}^{n_q} L_{i,j+1}^* s^j \right) \frac{\sum_{j=0}^{q_{i-1}} Z_{i-1,j+1} s^j}{\sum_{j=0}^{p_{i-1}} Y_{i-1,j+1} s^j} \\ &= \frac{\sum_{j=0}^{\bar{\delta}_q} \bar{\Delta}_{i,j+1} s^j \cdot \sum_{j=0}^{\hat{q}_{i-1}+n_q} \left(\sum_{k=0}^j d_{i,j-k+1}^* N_{i-1,k+1} - L_{i,j-k+1}^* Y_{i-1,k+1} \right) s^j \cdot \sum_{j=0}^{q_{i-1}} Z_{i-1,j+1} s^j}{\sum_{j=0}^{\bar{\delta}_q+2p_{i-1}} \left(\sum_{r=0}^j \sum_{k=0}^{j-r} \bar{\bar{\Delta}}_{i,j-k-r+1} Y_{i-1,k+1} Y_{i-1,r+1} \right) s^j} \\ &= \frac{\sum_{j=0}^{\bar{\delta}_q+q_{i-1}+\hat{q}_{i-1}+n_q} \sum_{t=0}^j \sum_{r=0}^{j-t} \sum_{k=0}^{j-t-r} \bar{\Delta}_{i,j-k-r-t+1} d_{i,k+1}^* N_{i-1,r+1} Z_{i-1,t+1} s^j}{\sum_{j=0}^{\bar{\delta}_q+2p_{i-1}} \left(\sum_{r=0}^j \sum_{k=0}^{j-r} \bar{\bar{\Delta}}_{i,j-k-r+1} Y_{i-1,k+1} Y_{i-1,r+1} \right) s^j} \\ &= \frac{\sum_{j=0}^{\bar{\delta}_q+q_{i-1}+\hat{q}_{i-1}+n_q} \sum_{t=0}^j \sum_{r=0}^{j-t} \sum_{k=0}^{j-t-r} \bar{\Delta}_{i,j-k-r-t+1} L_{i,k+1}^* Y_{i-1,r+1} Z_{i-1,t+1} s^j}{\sum_{j=0}^{\bar{\delta}_q+2p_{i-1}} \left(\sum_{r=0}^j \sum_{k=0}^{j-r} \bar{\bar{\Delta}}_{i,j-k-r+1} Y_{i-1,k+1} Y_{i-1,r+1} \right) s^j} \\ &= \frac{\sum_{j=0}^{\bar{b}_i} V_{i,j+1} s^j}{\sum_{j=0}^{\bar{\bar{b}}_i} W_{i,j+1} s^j}. \end{aligned}$$

It is not difficult to verify that in the last expression $V_{i,j+1}$ and $W_{i,j+1}$ satisfy (3.13) and (3.14), respectively.

Finally, using (2.4) of *Algorithm 2.1*, we obtain

$$\begin{aligned}
X_i(s) &= \begin{bmatrix} X_{i-1}(s) - (d_i(s) + \sigma_i(s)) b_i^*(s) \\ b_i^*(s) \end{bmatrix} \\
&= \begin{bmatrix} \frac{\sum_{j=0}^{q_i-1} Z_{i-1,j+1} s^j}{\sum_{j=0}^{p_i-1} Y_{i-1,j+1} s^j} - \left(\frac{\sum_{j=0}^{q_i-1+q} d_{i,j+1} s^j}{\sum_{j=0}^{p_i-1} Y_{i-1,j+1} s^j} + \frac{\sum_{j=0}^{q_i-1+\overline{n}_q+n_q} \varphi_{i,j+1} s^j}{\sum_{j=0}^{p_i-1+\overline{n}_q} \psi_{i,j+1} s^j} \right) \frac{\sum_{j=0}^{\overline{b}_i} V_{i,j+1} s^j}{\sum_{j=0}^{\overline{b}_i} W_{i,j+1} s^j} \\ \frac{\sum_{j=0}^{\overline{b}_i} V_{i,j+1} s^j}{\sum_{j=0}^{\overline{b}_i} W_{i,j+1} s^j} \end{bmatrix} \\
&= \begin{bmatrix} \frac{\sum_{j=0}^{q_i-1} Z_{i-1,j+1} s^j}{\sum_{j=0}^{p_i-1} Y_{i-1,j+1} s^j} - \frac{\sum_{j=0}^{q_i-1+q+\overline{n}_q} d_{i,j-k+1} \overline{N}_{i-1,k+1} s^j + \sum_{j=0}^{q_i-1+\overline{n}_q+n_q} \varphi_{i,j+1} s^j}{\sum_{j=0}^{p_i-1+\overline{n}_q} \psi_{i,k+1} s^j} \cdot \frac{\sum_{j=0}^{\overline{b}_i} V_{i,j+1} s^j}{\sum_{j=0}^{\overline{b}_i} W_{i,j+1} s^j} \\ \frac{\sum_{j=0}^{\overline{b}_i} V_{i,j+1} s^j}{\sum_{j=0}^{\overline{b}_i} W_{i,j+1} s^j} \end{bmatrix} \\
&= \begin{bmatrix} \frac{\sum_{j=0}^{q_i-1} Z_{i-1,j+1} s^j}{\sum_{j=0}^{p_i-1} Y_{i-1,j+1} s^j} - \frac{\left(\sum_{j=0}^{q_i-1+q+\overline{n}_q} d_{i,j-k+1} \overline{N}_{i-1,k+1} s^j + \sum_{j=0}^{q_i-1+\overline{n}_q+n_q} \varphi_{i,j+1} s^j \right) \frac{\sum_{j=0}^{\overline{b}_i} V_{i,j+1} s^j}{\sum_{j=0}^{\overline{b}_i} W_{i,j+1} s^j}}{\sum_{j=0}^{p_i-1+\overline{n}_q+\overline{b}_i} \sum_{k=0}^j \psi_{i,j-k+1} W_{i,k+1} s^j} \\ \frac{\sum_{j=0}^{\overline{b}_i} V_{i,j+1} s^j}{\sum_{j=0}^{\overline{b}_i} W_{i,j+1} s^j} \end{bmatrix} \\
&= \begin{bmatrix} \frac{\sum_{j=0}^{\theta_q} \Theta_{i,j+1} s^j}{\sum_{j=0}^{p_i-1+\overline{n}_q+\overline{b}_i} \sum_{k=0}^j \psi_{i,j-k+1} W_{i,k+1} s^j} \\ \frac{\sum_{j=0}^{\overline{b}_i} V_{i,j+1} s^j}{\sum_{j=0}^{\overline{b}_i} W_{i,j+1} s^j} \end{bmatrix} = \begin{bmatrix} \frac{\sum_{j=0}^{\theta_q} \Theta_{i,j+1} s^j}{\sum_{j=0}^{p_i-1+\overline{n}_q+\overline{b}_i} \sum_{k=0}^j \psi_{i,j-k+1} W_{i,k+1} s^j} \\ \frac{\sum_{j=0}^{\overline{b}_i} V_{i,j+1} s^j}{\sum_{j=0}^{\overline{b}_i} W_{i,j+1} s^j} \cdot \frac{\sum_{j=0}^{p_i-1+\overline{n}_q} \psi_{i,j+1} s^j}{\sum_{j=0}^{p_i-1+\overline{n}_q} \psi_{i,j+1} s^j} \end{bmatrix} \\
&= \begin{bmatrix} \frac{\sum_{j=0}^{\theta_q} \Theta_{i,j+1} s^j}{\sum_{j=0}^{p_i-1+\overline{n}_q+\overline{b}_i} \sum_{k=0}^j \psi_{i,j-k+1} W_{i,k+1} s^j} \\ \frac{\sum_{j=0}^{p_i-1+\overline{n}_q+\overline{b}_i} \sum_{k=0}^j \psi_{i,j-k+1} V_{i,k+1} s^j}{\sum_{j=0}^{p_i-1+\overline{n}_q+\overline{b}_i} \sum_{k=0}^j \psi_{i,j-k+1} W_{i,k+1} s^j} \end{bmatrix} = \frac{\sum_{j=0}^{q_i} Z_{i,j+1} s^j}{\sum_{j=0}^{p_i} Y_{i,j+1} s^j},
\end{aligned}$$

where $\Theta_{i,j+1}$ is defined in (3.19).

Finally, we obtain the polynomial representations for $Z_{i,j+1}$ and $Y_{i,j+1}$ as in (3.17)-(3.23)

In accordance with Lemma 2.1, the weighted Moore-Penrose inverse for given matrix is $A(s)_{M,N}^\dagger = X_n(s)$, which completes the proof. \square

The next algorithm is a generalization of *Algorithm 2.2* and computes the inverse matrix $N^{-1}(s)$ in a polynomial form.

Theorem 3.2. *Let the leading principal submatrix $N_i(s)$ of the positive definite matrix $N(s)$ is partitioned as in (2.2), and assume that $n_{ii}(s), l_i(s), N_{i-1}^{-1}(s)$ possesses the polynomial representation (3.5). Then the following algorithm computes the inverse matrix $N^{-1}(s)$.*

Algorithm 3.2. *Input: positive definite matrix $N(s)$.*

Step 1. Initial values:

$$\overline{N}_{1,j+1} = 1, \quad \overline{\overline{N}}_{1,j+1} = \widehat{n}_{1,j+1}, \quad 0 \leq j \leq \overline{n}_q. \quad (3.26)$$

Step 2. Recursive step: For $2 \leq i \leq n$ perform Step 2.1-Step 2.4:

Step 2.1. Compute

$$\overline{G}_{i,j+1} = \overline{\overline{N}}_{i-1,j+1}, \quad 0 \leq j \leq \overline{g}_q = \overline{\overline{n}}_q \quad (3.27)$$

$$p_{i,j+1} = \sum_{k=0}^j \widehat{n}_{i,j-k+1} \overline{\overline{N}}_{i-1,k+1}, \quad 0 \leq j \leq n_q + \overline{n}_q$$

$$q_{i,j+1} = \sum_{r=0}^j \sum_{k=0}^{j-r} L_{i,j-k-r+1}^* \overline{\overline{N}}_{i-1,k+1} L_{i,r+1}, \quad 0 \leq j \leq 2n_q + \overline{n}_q$$

$$\overline{\overline{G}}_{i,j+1} = p_{i,j+1} - q_{i,j+1}, \quad 0 \leq j \leq \overline{\overline{g}}_q = 2n_q + \overline{n}_q \quad (3.28)$$

where $p_{i,j+1}$ is padded by zeros from $n_q + \overline{n}_q$ up to upper bound $2n_q + \overline{n}_q$.

Step 2.2. Compute

$$\overline{F}_{i,j+1} = - \sum_{k=0}^j \overline{\overline{N}}_{i-1,j-k+1} L_{i,k+1}, \quad 0 \leq j \leq \overline{f}_q = \overline{n}_q + n_q \quad (3.29)$$

$$\overline{\overline{F}}_{i,j+1} = \overline{\overline{G}}_{i,j+1}, \quad 0 \leq j \leq \overline{\overline{f}}_q = \overline{\overline{g}}_q \quad (3.30)$$

Step 2.3. Compute

$$\begin{aligned} \overline{E}_{i-1,j+1} &= \sum_{r=0}^j \sum_{k=0}^{j-r} \overline{\overline{N}}_{i-1,j-k-r+1} \overline{G}_{i,k+1} \overline{\overline{F}}_{i,r+1} \\ &+ \overline{\overline{N}}_{i-1,j-k-r+1} \overline{F}_{i,k+1} \overline{F}_{i-1,r+1}^*, \end{aligned} \quad (3.31)$$

$$\begin{aligned} \overline{\overline{E}}_{i-1,j+1} &= \overline{\overline{N}}_{i-1,j-k-r+1} \overline{\overline{G}}_{i,k+1} \overline{\overline{F}}_{i-1,r+1}, \\ &0 \leq j \leq \overline{e}_q = \max(\overline{n}_q + \overline{g}_q + \overline{\overline{f}}_q, \overline{n}_q + 2\overline{f}_q), \\ &0 \leq j \leq \overline{\overline{e}}_q = \overline{\overline{n}}_q + \overline{g}_q + \overline{\overline{f}}_q. \end{aligned} \quad (3.32)$$

Step 2.4. Generate

$$\overline{N}_{i,j+1} = \begin{bmatrix} \sum_{r=0}^j \sum_{k=0}^{j-r} \overline{E}_{i,j-k-r+1} \overline{F}_{i,k+1} \overline{G}_{i,r+1} & \sum_{r=0}^j \sum_{k=0}^{j-r} \overline{E}_{i,j-k-r+1} \overline{F}_{i,k+1} \overline{G}_{i,r+1} \\ \sum_{r=0}^j \sum_{k=0}^{j-r} \overline{E}_{i,j-k-r+1} \overline{F}_{i,k+1}^* \overline{G}_{i,r+1} & \sum_{r=0}^j \sum_{k=0}^{j-r} \overline{E}_{i,j-k-r+1} \overline{F}_{i,k+1} \overline{G}_{i,r+1} \end{bmatrix} \quad (3.33)$$

$$0 \leq j \leq \overline{n}_q = \max\{\overline{g}_q + \overline{f}_q + \overline{e}_q, \overline{g}_q + \overline{f}_q + \overline{e}_q, \overline{g}_q + \overline{f}_q + \overline{e}_q, \overline{g}_q + \overline{f}_q + \overline{e}_q\}$$

$$\overline{\overline{N}}_{i,j+1} = \sum_{r=0}^j \sum_{k=0}^{j-r} \overline{\overline{E}}_{i,j-k-r+1} \overline{\overline{F}}_{i,k+1} \overline{\overline{G}}_{i,r+1}, \quad (3.34)$$

$$0 \leq j \leq \overline{\overline{n}}_q = \overline{\overline{g}}_q + \overline{\overline{f}}_q + \overline{\overline{e}}_q.$$

Step 3. Stopping criterion: for $i = n$ the inverse $N^{-1}(s) = N_n^{-1}(s)$ is equal to

$$N^{-1}(s) = \frac{\sum_{j=0}^{\overline{n}_q} \overline{N}_{n,j+1} s^j}{\sum_{j=0}^{\overline{\overline{n}}_q} \overline{\overline{N}}_{n,j+1} s^j}. \quad (3.35)$$

Proof. It is not difficult to verify that (3.26) follows from

$$N_1^{-1}(s) = n_{11}^{-1}(s) = \frac{1}{\sum_{j=0}^{n_q} \widehat{n}_{i,j+1} s^j}.$$

Also, (3.27), (3.28), (3.29), (3.30), (3.31) and (3.32) follows from the following. Using (2.15) we have

$$\begin{aligned} g_{ii}(s) &= \left(\sum_{j=0}^{\overline{g}_q} \overline{G}_{i,j+1} s^j \right) / \left(\sum_{j=0}^{\overline{\overline{g}}_q} \overline{\overline{G}}_{i,j+1} s^j \right) = (n_{ii}(s) - l_i^*(s) N_{i-1}^{-1}(s) l_i(s))^{-1} \\ &= \left(\sum_{j=0}^{n_q} \widehat{n}_{i,j+1} s^j - \sum_{j=0}^{n_q} L_{i,j+1}^* s^j \frac{\sum_{j=0}^{\overline{n}_q} \overline{N}_{i-1,j+1} s^j}{\sum_{j=0}^{\overline{\overline{n}}_q} \overline{\overline{N}}_{i-1,j+1} s^j} \sum_{j=0}^{n_q} L_{i,j+1} s^j \right)^{-1} \\ &= \frac{\sum_{j=0}^{\overline{\overline{n}}_q} \overline{\overline{N}}_{i-1,j+1} s^j}{\sum_{j=0}^{n_q + \overline{n}_q} \sum_{k=0}^j \widehat{n}_{i,j-k+1} \overline{\overline{N}}_{i-1,k+1} s^j - \sum_{j=0}^{2n_q + \overline{n}_q} \sum_{r=0}^j \sum_{k=0}^{j-r} L_{i,j-k-r+1}^* \overline{N}_{i-1,k+1} L_{i,r+1} s^j}. \end{aligned}$$

An application of (2.16) gives

$$\begin{aligned}
 f_i(s) &= \left(\sum_{j=0}^{\bar{f}_q} \bar{F}_{i,j+1} s^j \right) / \left(\sum_{j=0}^{\bar{f}_q} \bar{F}_{i,j+1} s^j \right) = -g_{ii}(s) N_{i-1}^{-1}(s) l_i(s) \\
 &= \frac{- \sum_{j=0}^{\bar{n}_q} \bar{G}_{i,j+1} s^j \sum_{j=0}^{\bar{n}_q} \bar{N}_{i-1,j+1} s^j}{\sum_{j=0}^{2n_q+\bar{n}_q} \bar{G}_{i,j+1} s^j \sum_{j=0}^{\bar{n}_q} \bar{N}_{i-1,j+1} s^j} \sum_{j=0}^{n_q} L_{i,j+1} s^j = \frac{- \sum_{j=0}^{\bar{n}_q+n_q} \sum_{k=0}^j \bar{N}_{i-1,j-k+1} L_{i,k+1} s^j}{\sum_{j=0}^{2n_q+\bar{n}_q} \bar{G}_{i-1,j+1} s^j}.
 \end{aligned}$$

In view of (2.17) one can verify the following:

$$\begin{aligned}
 E_{i-1}(s) &= \left(\sum_{j=0}^{\bar{e}_q} \bar{E}_{i,j+1} s^j \right) / \left(\sum_{j=0}^{\bar{e}_q} \bar{E}_{i,j+1} s^j \right) = N_{i-1}^{-1}(s) + g_{ii}^{-1}(s) f_i(s) f_i^*(s) \\
 &= \frac{\sum_{j=0}^{\bar{n}_q} \bar{N}_{i-1,j+1} s^j}{\sum_{j=0}^{\bar{n}_q} \bar{N}_{i-1,j+1} s^j} + \frac{\sum_{j=0}^{\bar{g}_q} \bar{G}_{i,j+1} s^j \sum_{j=0}^{\bar{f}_q} \bar{F}_{i,j+1} s^j \sum_{j=0}^{\bar{f}_q} \bar{F}_{i,j+1}^* s^j}{\sum_{j=0}^{\bar{n}_q} \bar{N}_{i-1,j+1} s^j \sum_{j=0}^{\bar{g}_q} \bar{G}_{i,j+1} s^j \sum_{j=0}^{\bar{f}_q} \bar{F}_{i,j+1} s^j \sum_{j=0}^{\bar{f}_q} \bar{F}_{i,j+1}^* s^j} \\
 &= \frac{\sum_{j=0}^{\max(\bar{n}_q+\bar{g}_q+2\bar{g}_q, \bar{n}_q+\bar{g}_q+2\bar{f}_q)} \sum_{r=0}^j \sum_{k=0}^{j-r} \bar{N}_{i-1,j-k-r+1} \bar{G}_{i,k+1} \bar{F}_{i,r+1}}{\sum_{j=0}^{\bar{n}_q+\bar{g}_q+\bar{f}_q} \sum_{r=0}^j \sum_{k=0}^{j-r} \bar{N}_{i-1,j-k-r+1} \bar{G}_{i,k+1} \bar{F}_{i,r+1} s^j} \\
 &\quad + \frac{\sum_{j=0}^{\max(\bar{n}_q+\bar{g}_q+2\bar{g}_q, \bar{n}_q+\bar{g}_q+2\bar{f}_q)} \bar{N}_{i-1,j-k-r+1} \bar{F}_{i,k+1}^* \bar{F}_{i,r+1} s^j}{\sum_{j=0}^{\bar{n}_q+\bar{g}_q+\bar{f}_q} \sum_{r=0}^j \sum_{k=0}^{j-r} \bar{N}_{i-1,j-k-r+1} \bar{G}_{i,k+1} \bar{F}_{i,r+1} s^j}.
 \end{aligned}$$

Using (2.2) we finally get the inverse

$$\begin{aligned}
 N_i^{-1}(s) &= \begin{bmatrix} E_{i-1}(s) & f_i(s) \\ f_i^*(s) & g_{ii}(s) \end{bmatrix} = \begin{bmatrix} \frac{\sum_{j=0}^{\bar{e}_q} \bar{E}_{i,j+1} s^j}{\sum_{j=0}^{\bar{e}_q} \bar{E}_{i,j+1} s^j} & \frac{\sum_{j=0}^{\bar{f}_q} \bar{F}_{i,j+1} s^j}{\sum_{j=0}^{\bar{f}_q} \bar{F}_{i,j+1} s^j} \\ \frac{\sum_{j=0}^{\bar{f}_q} \bar{F}_{i,j+1}^* s^j}{\sum_{j=0}^{\bar{f}_q} \bar{F}_{i,j+1} s^j} & \frac{\sum_{j=0}^{\bar{g}_q} \bar{G}_{i,j+1} s^j}{\sum_{j=0}^{2n_q+\bar{n}_q} \bar{G}_{i,j+1} s^j} \end{bmatrix} \\
 &= \begin{bmatrix} \frac{\sum_{j=0}^{\bar{g}_q+\bar{f}_q+\bar{e}_q} \sum_{r=0}^j \sum_{k=0}^{j-r} \bar{E}_{i,j-k-r+1} \bar{F}_{i,k+1} \bar{G}_{i,r+1}}{\sum_{j=0}^{\bar{g}_q+\bar{f}_q+\bar{e}_q} \sum_{r=0}^j \sum_{k=0}^{j-r} \bar{E}_{i,j-k-r+1} \bar{F}_{i,k+1} \bar{G}_{i,r+1}} & \frac{\sum_{j=0}^{\bar{g}_q+\bar{f}_q+\bar{e}_q} \sum_{r=0}^j \sum_{k=0}^{j-r} \bar{E}_{i,j-k-r+1} \bar{F}_{i,k+1} \bar{G}_{i,r+1}}{\sum_{j=0}^{\bar{g}_q+\bar{f}_q+\bar{e}_q} \sum_{r=0}^j \sum_{k=0}^{j-r} \bar{E}_{i,j-k-r+1} \bar{F}_{i,k+1} \bar{G}_{i,r+1}} \\ \frac{\sum_{j=0}^{\bar{g}_q+\bar{f}_q+\bar{e}_q} \sum_{r=0}^j \sum_{k=0}^{j-r} \bar{E}_{i,j-k-r+1} \bar{F}_{i,k+1} \bar{G}_{i,r+1}}{\sum_{j=0}^{\bar{g}_q+\bar{f}_q+\bar{e}_q} \sum_{r=0}^j \sum_{k=0}^{j-r} \bar{E}_{i,j-k-r+1} \bar{F}_{i,k+1} \bar{G}_{i,r+1}} & \frac{\sum_{j=0}^{\bar{g}_q+\bar{f}_q+\bar{e}_q} \sum_{r=0}^j \sum_{k=0}^{j-r} \bar{E}_{i,j-k-r+1} \bar{F}_{i,k+1} \bar{G}_{i,r+1}}{\sum_{j=0}^{\bar{g}_q+\bar{f}_q+\bar{e}_q} \sum_{r=0}^j \sum_{k=0}^{j-r} \bar{E}_{i,j-k-r+1} \bar{F}_{i,k+1} \bar{G}_{i,r+1}} \end{bmatrix}
 \end{aligned}$$

which confirms (3.33), (3.34) and (3.35). \square

4 Examples

Example 4.1. Find the weighted Moore-Penrose inverse of the rational matrix

$$X(s) = \{\{s+1, s+2, s\}, \{s, s, s+1\}, \{s+1, s+2, s\}\}$$

using the following weighting matrices, $M_1(s)$ and $N_1(s)$:

$$M_1(s) = \{\{s+1, s, s+1\}, \{s, s+2, s\}, \{s+1, s, s+3\}\};$$

$$N_1(s) = \{\{s+1, s+1, s+1\}, \{s+1, s+2, s\}, \{s+1, s, s+3\}\};$$

The following result is generated applying the function *WPartit*, implementing Algorithm 2.1 (see implementation details):

`WPartit[X, M1, N1]`

$$\text{WEIGHTED MOORE-PENROSE INVERSE} = \begin{pmatrix} \frac{2s^2(2+s)}{12+32s+33s^2+14s^3} & -\frac{(2+s)^2}{2+3s+2s^2} & \frac{s(12+16s+5s^2)}{12+32s+33s^2+14s^3} \\ \frac{2+5s+2s^2}{(6+7s)(2+3s+2s^2)} & \frac{2(1+s)}{2+3s+2s^2} & \frac{4+2s-2s^2}{12+32s+33s^2+14s^3} \\ -\frac{2s^2(1+s)}{(6+7s)(2+3s+2s^2)} & \frac{(1+s)(2+s)}{2+3s+2s^2} & -\frac{s(1+s)(6+5s)}{(6+7s)(2+3s+2s^2)} \end{pmatrix}.$$

Example 4.2. In this example we compute the weighted Moore-Penrose inverse of the rational matrix $X(s)$ due to the following weights $M_1(s)$ and $N_1(s)$:

$$X = \{\{1/s^2, s, (s+1)/s^3\}, \{s, s^2-1, s\}, \{s+1, 1/s, s+1\}\};$$

$$M_1 = \{\{s+1, s, s+1\}, \{s, s+2, s\}, \{s+1, s, s+3\}\};$$

$$N_1 = \{\{s+1, s+1, s+1\}, \{s+1, s+2, s\}, \{s+1, s, s+3\}\};$$

`WPartit[X, M1, N1]`

$$\text{WEIGHTED MOORE-PENROSE INVERSE} = \begin{pmatrix} -s^3 & \frac{-1-s+s^5+s^6}{s(-2-s+s^2+s^3)} & \frac{-1-s+s^2+s^3-s^5}{-2-s+s^2+s^3} \\ 0 & \frac{1+s}{-2-s+s^2+s^3} & \frac{s}{2+s-s^2-s^3} \\ s^3 & -\frac{-1+s^4+s^5}{-2-s+s^2+s^3} & \frac{s-s^3+s^5}{-2-s+s^2+s^3} \end{pmatrix}.$$

Example 4.3. If the matrices are considered in the polynomial form, then the function *WPartPoly*, implementing Algorithm 3.1, can be used to compute the weighted Moore-Penrose inverse of the matrix X (see implementation details):

$$X = \{\{1+s, -2+s^4, s\}, \{s, -1+s, s\}, \{s, s, 1+s\}\}; \quad M_1 = N_1 = \{\{1+s, s, s\}, \{s, -1+s, s\}, \{s, s, 1+s\}\};$$

`WPartPoly[X, M1, N1]`

$$\text{WEIGHTED MOORE-PENROSE INVERSE} = \begin{pmatrix} \frac{1}{1-s-s^2+s^5} & \frac{2+2s+s^2-s^4-s^5}{-1+s+s^2-s^5} & \frac{s+s^2-s^5}{1-s-s^2+s^5} \\ \frac{s}{1-s-s^2+s^5} & \frac{1+2s}{-1+s+s^2-s^5} & \frac{s}{1-s-s^2+s^5} \\ \frac{s}{-1+s+s^2-s^5} & \frac{s(3+s-s^4)}{1-s-s^2+s^5} & \frac{-1+2s+s^2-s^5}{-1+s+s^2-s^5} \end{pmatrix}.$$

Example 4.4. In this example we generate the Moore-Penrose inverse of the matrix $X(s)$, known as the parameter test matrix of Hessenberg form [32]:

$$X = \{\{s, 1, 0, 0, 0\}, \{s^2, s, 1, 0, 0\}, \{s^3, s^2, s, 1, 0\}, \{s^4, s^3, s^2, s, 1\}, \{s^5, s^4, s^3, s^2, s\}\}$$

using identity matrices $M_1(s)$ and $N_1(s)$ of appropriate orders, we get:

`WPartPoly[X, IdentityMatrix[5], IdentityMatrix[5]]`

$$\text{WEIGHTED MOORE-PENROSE INVERSE} = \begin{pmatrix} \frac{s}{(1+s)^2} & 0 & 0 & 0 & 0 \\ \frac{1}{(1+s)^2} & 0 & 0 & 0 & 0 \\ -s & 1 & 0 & 0 & 0 \\ 0 & -s & 1 & 0 & 0 \\ 0 & 0 & -s & \frac{1}{(1+s)^2} & \frac{s}{(1+s)^2} \end{pmatrix}.$$

5 Conclusion

We extend Wang's partition method from [28] to the set of one-variable rational and polynomial matrices. In this way, we obtain an algorithm for symbolic computation of the weighted Moore-Penrose inverse of one-variable rational and polynomial matrices. The paper is a generalization of the paper [28] and a continuation of the paper [23]. Several symbolic examples are arranged. In partial case $M = I_m$, $N = I_n$ we obtain the usual Moore-Penrose inverse, and then use test examples from [32]. Main implementation details are described as the appendix in the next section.

6 Implementation details

For the sake of completeness we describe the MATHEMATICA code which implements Algorithm 2.1. and Algorithm 3.1.

6.1 Rational matrix case

Main problem in the implementation of Algorithm 2.1 is the simplification of algebraic expressions included. This difficulty imposes its implementation in a symbolic computational package. Moreover, a significant problem in the implementation of *Algorithm 2.1* is the magnification of arithmetic operations. This problem increased by multiplicative recomputations. In view of *Step 2* in *Algorithm 2.1*, for each $i \in \{2, \dots, n\}$, the Moore-Penrose inverse $X_i(s)$ must be computed $n - i + 1$ times. Moreover, in view of *Step 2.1* and *Step 2.3*, the pseudoinverse $X_{i-1}(s)$ is needful during the computation of the values $d_i(s)$ and $b_i(s)$. Consequently, *Algorithm 2.1* requires $3(n - i + 1)$ recomputations of the Moore-Penrose inverse $X_i(s)$, for each $i \in \{2, \dots, n\}$. The total number of different values that will be produced is comparatively small, but these values must be recomputed many times by means of relatively complicated expressions. In order to obviate recomputations, we use possibility of the programming package MATHEMATICA to define functions that remember values they have found [31, 30]. The pattern for defining a memo function is $f[x_]:=f[x]=rhs$.

In order to enable simplifications of rational expressions by means of MATHEMATICA function *Simplify*, we restrict our implementation to the set of rational matrices with real coefficients.

In the beginning we describe two auxiliary procedures.

A. The function $Col[a, j]$ extracts j -th column of the matrix $a = A(s)$:

```
Col[a_List, j_] := Transpose[{Transpose[a][[j]]}]
```

B. The submatrix $\hat{A}_j(s) = [a_1(s), \dots, a_j(s)]$ which contains first $j \leq n$ columns of the matrix $A(s) = \hat{A}_n(s) = [a_1(s), \dots, a_n(s)]$ is generated as follows:

```
Adop[a_List, j_] := Module[{m, n},
  {m, n} = Dimensions[a];
  Return[Transpose[Drop[Transpose[a], -(n-j)]]];]
```

Step 2 of the Algorithm 2.1 is implemented in the following functions which remember before computed values.

Implementation of Step 2.1.

```
DD[a_List,m0_List,n0_List,i_]:=DD[a,m0,n0,i]=
Module[{s={}},
s=Simplify[A[a,m0,n0,i-1].Col[a,i]];
Return[s]]
```

Implementation of Step 2.2.

```
CC[a_List,m0_List,n0_List,i_]:=CC[a,m0,n0,i]=
Module[{s={}},
s=Col[a,i]-Adop[a,i-1].DD[a,m0,n0,i];
Return[Simplify[s]]]
```

Implementation of Step 2.3.

```
B[a_List,m0_List,n0_List,i_]:=B[a,m0,n0,i]=
Module[{nul,m1,j,k,n1,s={}},
{m1,n1}=Dimensions[CC[a,m0,n0,i]];
nul=Table[0,{j,1,m1},{k,1,n1}];
If[CC[a,m0,n0,i]!=nul,
s=Inverse[Transpose[CC[a,m0,n0,i]].m0.CC[a,m0,n0,i]]
.Transpose[CC[a,m0,n0,i]].m0,
s=(Delt[a,m0,n0,i])^(-1).(Transpose[DD[a,m0,n0,i]].NK[n0,i][[1]]
-Transpose[NK[n0,i][[3]]]).A[a,m0,n0,i-1]];
Return[Simplify[s]]]
```

The following function $Delt[a, m0, n0, i]$ computes δ_i defined in (2.8).

```
Delt[a_List,m0_List,n0_List,i_]:=Module[{s},
s=NK[n0,i][[2]]+Transpose[DD[a,m0,n0,i]].NK[n0,i][[1]].DD[a,m0,n0,i] -
(Transpose[DD[a,m0,n0,i]].NK[n0,i][[3]]+Transpose[NK[n0,i][[3]]].DD[a,m0,n0,i])
-Transpose[NK[n0,i][[3]]].(IdentityMatrix[i-1]-A[a,m0,n0,i-1].Adop[a,i-1])
.Inverse[NK[n0,i][[1]]].NK[n0,i][[3]];
Return[Simplify[s]]]
```

In the function $NK[a, i]$ we find the partition (2.2) of the leading principal submatrix $N_i(s)$ of the weighted matrix $N(s)$.

```
NK[a_List,i_]:=Module[{lk,NK1,nkk},
nkk={a[[i,i]]};
If[i==1,Return[{nkk,nkk,nkk}],
NK1=Transpose[Take[Transpose[Take[a,i-1]],i-1]];
lk =Transpose[{Most[Last[Take[Transpose[Take[a,i]],i]]]}];
Return[{NK1,nkk,lk}]]]
```

Implementation of Step 1 and Step 2.4.

```
A[a_List,m0_List,n0_List,i_]:=A[a,m0,n0,i]=
Module[{b=a},
If[i==1, (* Compute X1(s) *)
If[Col[a,i]==Col[a,i]*0,
b=Transpose[a][[1]], (* a1(s)=0 *)
b=Inverse[{Transpose[a][[i]].m0.Col[a,i]},{Transpose[a][[1]].m0}], (* a1(s)!=0 *)
(* Compute Xi(s), i>1 *)
b=A[a,m0,n0,i-1]-(DD[a,m0,n0,i]+(IdentityMatrix[i-1]-A[a,m0,n0,i-1].Adop[a,i-1])
.Inverse[NK[n0,i][[1]]].NK[n0,i][[3]]).B[a,m0,n0,i];
b=Append[b,B[a,m0,n0,i][[1]]];
Return[Simplify[b]]]
```

The following function starts recursive computations in *Step 2*:

```
WPartit[a_List,m0_List,n0_List]:=
Module[{m,n,i},{m,n}=Dimensions[a];
Print["WEIGHTED MOORE-PENROSE INVERSE="];
A[a,m0,n0,n] // MatrixForm]

6.2 Polynomial matrix case

We also restrict the implementation to the set of polynomial matrices with
real coefficients. The matrix  $A(s)$  defined in (3.1) can be represented as the
list  $\{A_1, \dots, A_{q+1}\}$ . The  $i$ -th column  $a_i(s)$  of  $A(s)$  is the polynomial matrix
defined in (3.3), and therefore can be represented by the three-dimensional list
 $\{a_{i,1}, \dots, a_{i,q+1}\}$ ,  $1 \leq i \leq n$ .

Col[L_List,j_]:= (* Compute j-th column from L *)
Module[{L1=L2={},i},
For[i=1,i<=Length[L],i++,
L1=Append[L1,Transpose[L[[i]]]]; AppendTo[L2,Transpose[{L1[[i,j]]}]]];
Return[L2]];

FrmPoly[M_List]:= (* Form the polynomial matrix of the form (3.1) *)
Module[{L={},i,M1=M,v,s},
v=Variables[M];
If[v!={},
s=v[[1]]; (* The matrix is not constant *)
For[i=1,i<=Max[Exponent[M,s]],i++,
AppendTo[L,Coefficient[M,s^i]]; M1=M1-Coefficient[M,s^i]*s^i];
M1={M1};
For[i=1,i<=Length[L],i++, AppendTo[M1,L[[i]]]]];
If[v!={},Return[Simplify[M1]], (* The matrix is not constant *)
Return[Simplify[{M1}]]] (* The matrix is constant *) ];

TakeFPoly[L_List,j_]:= (* Separate first j columns from L *)
Block[{L1={},i},
For[i=1,i<=Length[L],i++, L1=Append[L1,Take[Transpose[L[[i]]],j]]];
Return[L1]];

DopZero[L_List,i_]:= (* Complete the matrix L by zero rows *)
Module[{L1=L,j,nula},
nula=L1[[1]]*0;
For[j=1,j<=i-Length[L],j++, AppendTo[L1,nula]];
Return[L1]];

LastZeroP[L_List]:= (* Drop the last zero rows from L *)
Module[{L1=L,Us=True,nul,d1},
If[L1!={},
While[Us && L1!={}, d1=Dimensions[L1][[1]];
If[L1[[d1]]==L1[[d1]]*0, L1=Drop[L1,-1], Us=False]];
Return[L1]];

DDP[L_List,M_List,N_List,i_]:=DDP[L,M,N,i]= (* Compute d_{i,j+1} using (3.8) *)
Module[{Y={},gr=0,bb,NN,L2={},L1=L,j,nula={}},
L2=ZZP[L,M,N,i-1]; gr=Length[L]+Length[L2];
nula={Table[0,{j,1,gr}]}; L1=DopZero[L1,gr]; NN=Col[L1,i]; L2=DopZero[L2,gr];
For[j=0,j<gr-1,j++,
If[(j+1)>Length[Y],Y=Join[Y,nula]];
Y[[j+1]]=Sum[L2[[j-k+1]].NN[[k+1]],{k,0,j}];
];
Y=LastZeroP[Y];Return[Y]];

CCP[L_List,M_List,N_List,i_]:=CCP[L,M,N,i]= (* Compute c_{i,j+1} using (3.9) *)
Module[{Y=L4={},gr=0,NN,L1=L,L2=L3={},j,nula={}},
L2=YYP[L,M,N,i-1]; gr=2Length[L]+Length[L2]; nula={Table[0,{j,1,gr}]}];
```

```

L1=DopZero[L1,gr]; NN=Col[L1,i]; L2=DopZero[L2,gr]; L4=DDP[L,M,N,i];
If[L4=={},L4={0}]; L4=DopZero[L4,gr]; L3=TakeFPoly[L,i-1]; L3=DopZero[L3,gr];
For[j=0,j<gr-1,j++,
  If[(j+1)>Length[Y],Y=Join[Y,nula];
  If[(Length[L4[[1]]]==0),
    Y[[j+1]]=Sum[NN[[j-k+1]]L2[[k+1]]-(L3[[j-k+1]]L4[[k+1]])[[1]],{k,0,j}],
    Y[[j+1]]=Sum[NN[[j-k+1]]L2[[k+1]]-Transpose[L3[[j-k+1]]].L4[[k+1]],{k,0,j}]]];
Return[LastZeroP[Y]]];

VVP[L_List,M_List,N_List,i_]:=VVP[L,M,N,i]=(* Compute V_{i,j+1} using (3.11) *)
Module[{M0=M,L1={},L2={},L3={},L4={},L5={},L6={},L7={},
  L8={},Y={},j,k,r,iz,q,mq,q1,gr},
  L2=CCP[L,M,N,i]; L5=DDP[L,M,N,i]; L6=NKP[N,i][[1]];
  L7=NKP[N,i][[3]]; L8=ZZP[L,M,N,i-1];
  If[L2!={},L1=YVP[L,M,N,i-1];
  mq=Length[M0]-1; q=Length[L]-1; q1=Length[L1];
  gr=q+2*Length[L1]+mq; L1=DopZero[L1,gr]; L2=DopZero[L2,gr];
  M0=DopZero[M0,gr]; iz={};
  For[j=0,j<gr,j++,
    iz=Join[iz,{Sum[Sum[Sum[L1[[j-k-r+1]]Transpose[L2[[k+1]]].
      M0[[r+1]],{k,0,j-r}],{r,0,j}]][[1]]}],
  (* Else *)
  iz={};L4=Delt[L,M,N,i][[1]];
  gr=Length[L4]-1+2*Length[XPP[L,M,N,i-1]]+Length[L]-1+Length[N]-1;
  L1=YVP[L,M,N,i-1]; L1=DopZero[L1,gr]; L4=DopZero[L4,gr];
  L5=DopZero[L5,gr]; L6=DopZero[L6,gr]; L7=DopZero[L7,gr];
  L8=DopZero[L8,gr]; Y={};
  For[j=0,j<gr,j++,
    If[Length[Dimensions[L5[[1]]]]==1,
      Y=Join[Y,Sum[L5[[j-k+1]]L6[[k+1]]
        -Transpose[L7[[j-k+1]]]L1[[k+1]],{k,0,j}]],
      Y=Join[Y,Sum[Transpose[L5[[j-k+1]]].L6[[k+1]]
        -Transpose[L7[[j-k+1]]]L1[[k+1]],{k,0,j}]]];
  Y=DopZero[Y,gr];
  For[j=0,j<gr,j++,
    If[Length[Dimensions[L8[[1]]]]==1,
      iz=Join[iz,{Sum[Sum[L4[[j-k-r+1,1]]Y[[k+1]].
        {L8[[r+1]],{k,0,j-r}],{r,0,j}]]}],
      iz=Join[iz,{Sum[Sum[L4[[j-k-r+1,1]]Y[[k+1]].
        L8[[r+1]],{k,0,j-r}],{r,0,j}]]}],
  Return[LastZeroP[iz]]];

WWP[L_List,M_List,N_List,i_]:=WWP[L,M,N,i]=(* Compute W_{i,j+1} using (3.12) *)
Module[{Y={},M0=M,gr=0,iz,L0={},L1={},L2={},L3={},L4={},j,nula={},mq},
  L2=CCP[L,M,N,i]; L3=YVP[L,M,N,i-1]; L0=Delt[L,M,N,i][[2]];
  If[L2!={},iz={};gr=2*Length[L2]+Length[M0]-1;
  L2=DopZero[L2,gr]; M0=DopZero[M0,gr];
  For[j=0,j<gr,j++,
    iz=Join[iz,Sum[Sum[Transpose[L2[[j-k-r+1]]].M0[[k+1]].L2[[r+1]],{k,0,j-r}],{r,0,j}]]],
  gr=2*Length[L3]+Length[L0]-1;
  L4=Transpose[{Delt[L,M,N,i][[2]]}];
  iz={};gr=Length[L4]-1+2*Length[L3];
  L3=DopZero[L3,gr]; L4=DopZero[L4,gr];
  For[j=0,j<gr,j++,
    iz=Join[iz,Sum[Sum[L4[[j-k-r+1]]L3[[k+1]]L3[[r+1]],{k,0,j-r}],{r,0,j}]]];
  Return[LastZeroP[iz]]];

ZZP[L_List,M_List,N_List,i_]:=ZZP[L,M,N,i]=(* Compute Z_{i,j+1} using (3.17) and (3.6) *)
Module[{L1=L,L2={},L3={},L4={},L5={},L6={},L7={},L8={},
  M0=M,mq,rez,NN1,q,gr,gr2,iz,iz1,j,k,r},
  If[i==1,(* Step 1 *)
  mq=Length[M0]-1;q=Length[L1]-1;L2={};
  For[j=1,j<Length[Col[L1,1]],j++,
    L2=Join[L2,Transpose[Col[L1,1][[j]]]]];
  If[LastZeroP[L2]==={},rez=L2,
  L2=DopZero[L2,mq+q+1]; M0=DopZero[M0,mq+q+1]; iz={};
  For[j=0,j<q+mq+1,j++,

```

```

    iz=Join[iz,{Sum[Sum[L2[[j-k+1]].M0[[k+1]],{k,0,j}]]]};
    rez=LastZeroP[iz];If[rez=={},rez={L2[[1]].M0[[1]]*0}],
(*Else *)
    L4=VVP[L,M,N,i]; iz=TET[L,M,N,i]; L7=KSIP[L,M,N,i];
    gr2=Length[iz]+2; iz=DopZero[iz,gr2];
    L4=DopZero[L4,gr2]; L7=DopZero[L7,gr2]; iz1={};
    For[j=0,j<gr2,j++,
        iz1=Join[iz1,{Sum[L7[[j-k+1,1]]L4[[k+1]],{k,0,j}]]];
    rez={};
    If[LastZeroP[iz]==={},iz=iz1*0];
    For[j=0,j<gr2,j++,
        If[Length[Dimensions[iz[[1]]]] == 1,
            rez=Join[rez,{Join[{iz[[j+1]]},{iz1[[j+1]]}]}],
            If[Dimensions[iz[[1]]][[1]] == 1,
                rez=Join[rez,{Join[iz[[j+1]],{iz1[[j+1]]}]}],
                rez=Join[rez,{Join[iz[[j+1]],{iz1[[j+1]]}]}]
            ]];(* EndIF *)
    Return[rez];

YYP[L_List,M_List,N_List,i_]:=YYP[L,M,N,i]= (* Compute  $Y_{i,j+1}$  using (3.18) and (3.7) *)
Module[{L1=L,L2={},L3={},L4={},L5={},M0=M,iz={},q,mq,j,k,r,gr},
    If[i==1, mq=Length[M0]-1;q=Length[L1]-1;L2={};
        For[j=1,j<=Length[Col[L1,1]],j++,
            L2=Join[L2,Transpose[Col[L1,1][[j]]]]];
        If[LastZeroP[L2]==={},iz=L2,
            L2=DopZero[L2,mq+2*q+1]; M0=DopZero[M0,mq+2*q+1];
            L3=DopZero[Col[L1,1],mq+2*q+1]; iz={};
            For[j=0,j<2*q+mq+1,j++,
                iz=Join[iz,Sum[Sum[Sum[L2[[j-k-r+1]].M0[[k+1]].L3[[r+1]],{k,0,j-r}]],{r,0,j}]]];
            iz=LastZeroP[iz];
        (* Else *)
        L3=KSIP[L,M,N,i]; L5=WVP[L,M,N,i];
        gr=Length[L3]+Length[L5]-1;
        L5=DopZero[L5,gr]; L3=DopZero[L3,gr]; iz={};
        For[j=0,j<gr,j++,
            iz=Join[iz,Sum[L3[[j-k+1]]L5[[k+1]],{k,0,j}]]];
        iz=LastZeroP[iz];
    Return[iz];

NKP[L_List,i_]:=NKP[L,i]= (* Find the partition (2.2) *)
Module[{lk={},NK1={},nkk={},L1={},L2={},L3={},L4={},L5={}},
    For[j=0,j<Length[L],j++,nkk=Join[nkk,{L[[j+1]][[i,i]]}]];
    If[i==1,Return[{nkk,nkk,nkk}],
        For[j=0,j<Length[L],j++,L1=Join[L1,{Take[L[[j+1]],i-1]}]];
        For[j=0,j<Length[L],j++,L2=Join[L2,{Transpose[L1[[j+1]]]}]];
        For[j=0,j<Length[L],j++,NK1=Join[NK1,{Take[L2[[j+1]],i-1]}]];
        L5={};L4={};L3={};L2={};L1={};
        For[j=0,j<Length[L],j++,L1=Join[L1,{Take[L[[j+1]],i]}]];
        For[j=0,j<Length[L],j++,L2=Join[L2,{Transpose[L1[[j+1]]]}]];
        For[j=0,j<Length[L],j++,L3=Join[L3,{Take[L2[[j+1]],i]}]];
        For[j=0,j<Length[L],j++,L4=Join[L4,{Last[L3[[j+1]]]}]];
        For[j=0,j<Length[L],j++,L5=Join[L5,{Most[L4[[j+1]]]}]];
        For[j=0,j<Length[L],j++,lk=Join[lk,{Transpose[{L5[[j+1]]}]}]];
    ];
    Return[{NK1,nkk,lk}];

KSIP[L_List,M_List,N_List,i_]:=KSIP[L,M,N,i]= (* Compute  $\psi_{i,j+1}$  (3.21) *)
Module[{L1={},L2={},L3={},L4={},L5={},iz,gr,j},
    L1=YYP[L,M,N,i-1]; L2=ZZP[L,M,N,i-1]; L3=InvNKP[N,i];
    L4=L3[[3]]; gr=Length[L1]+L3[[4]]; iz={};
    L1=DopZero[L1,gr]; L4=DopZero[L4,gr];
    For[j=0,j<gr,j++,
        iz=Join[iz,{Sum[L1[[j-k+1]]L4[[k+1]],{k,0,j}]}];
    iz=LastZeroP[iz];Return[iz];

FIP[L_List,M_List,N_List,i_]:=FIP[L,M,N,i]= (* Compute  $\varphi_{i,j+1}$  (3.20) *)
Module[{L1={},L2={},L3={},L4={},L5={},L6={},iz,iz1,Y,Y1,gr,gr1,gr2,gr3,j},
    L1=YYP[L,M,N,i-1]; L2=ZZP[L,M,N,i-1]; L3=InvNKP[N,i]; L4=L3[[1]];

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L6=NKP[N,i][[3]]; L5=TakeFPoly[L,i-1];
gr1=Max[Length[L1]-1,Length[L2]+Length[L]-1];
L0=DopZero[FrmPoly[IdentityMatrix[i-1]],gr1];
L1=DopZero[L1,gr1];Y1={};
For[j=0,j<gr1,j++,
  Y1=Join[Y1,{Sum[L1[[j-k+1]]L0[[k+1]],{k,0,j}}]];
gr2=Length[L2]+Length[L]-1;
L2=DopZero[L2,gr2];L5=DopZero[L5,gr2];Y={};
For[j=0,j<gr2,j++,
  If[Length[Dimensions[L2[[1]]]]==1,
    Y=Join[Y,{Sum[{L2[[j-k+1]]}.Transpose[L5[[k+1]]],{k,0,j}}]},
    Y=Join[Y,{Sum[L2[[j-k+1]].Transpose[L5[[k+1]]],{k,0,j}}]}
  ];
gr3=L3[[2]]+Length[N];L4=DopZero[L4,gr3];
L6=DopZero[L6,gr3]; iz1={};
For[j=0,j<gr3,j++,iz1=Join[iz1,{Sum[L4[[j-k+1]].L6[[k+1]],{k,0,j}}]];
gr=gr1+gr3; Y=DopZero[Y,gr]; Y1=DopZero[Y1,gr];
iz1=DopZero[iz1,gr]; iz={};
For[j=0,j<gr,j++,
  iz=Join[iz,{Sum[(Y1[[j-k+1]]-Y[[j-k+1]]) . iz1[[k+1]],{k,0,j}}]];
iz=LastZeroP[iz];Return[iz];

TET[L_List,M_List,N_List,i_]:=TET[L,M,N,i]=(* Compute \Theta_{i,j+1} (3.19) *)
Module[{L1={},L2={},L3={},L4={},L5={},L6={},iz,iz1,iz2,iz3, gr,gr1,j,k1,k2},
  L1=ZZP[L,M,N,i-1]; L2=InvNKP[N,i];gr1=L2[[4]];
  L2=L2[[3]]; L3=WVP[L,M,N,i]; L4=VVP[L,M,N,i];
  L5=DDP[L,M,N,i];L6=FIP[L,M,N,i]; iz={};
  gr=Length[L1]+gr1+Length[L3]+Length[L4]-1;
  L1=DopZero[L1,gr]; L2=DopZero[L2,gr]; L3=DopZero[L3,gr]; L4=DopZero[L4,gr];
  If[L5=={},For[j=1,j<=i,j++,L5=Append[L5,{0}]]];
  L5=DopZero[L5,gr]; iz1={};iz2={};iz3={};
  For[j=0,j<gr,j++,
    iz1=Join[iz1,{Sum[Sum[L1[[j-k-r+1]]L2[[k+1,1]]L3[[r+1]],{k,0,j-r}},{r,0,j}}]];
    If[L5[[1]]*0==={0},
      iz2=Join[iz2,{Sum[Sum[{L5[[j-k-r+1]]L2[[k+1,1]]}.L4[[r+1]],{k,0,j-r}},{r,0,j}}]},
      iz2=Join[iz2,{Sum[Sum[(L5[[j-k-r+1]]L2[[k+1,1]]).L4[[r+1]],{k,0,j-r}},{r,0,j}}]]];
  If[L6=={},L6=Table[{0},{k1,1},{k2,i-1}]];
  L6=DopZero[L6,gr];
  For[j=0,j<gr,j++,
    iz3=Join[iz3,{Sum[L6[[j-k+1]].L4[[k+1]],{k,0,j}}]];
  For[j=0,j<gr,j++,
    If[i==2,iz=Join[iz,{iz1[[j+1]]-iz2[[j+1]]-iz3[[j+1]]}],
      iz=Join[iz,{iz1[[j+1]]-iz2[[j+1]]-iz3[[j+1]]}]];
  iz=LastZeroP[iz]; If[iz=={},iz={0}];
  Return[iz];

WPartPoly[L_List,M_List,N_List]:=(* Implementation of Algorithm 3.1 *)
Module[{mm,nn,k,rez={},L1={},L2={},L3={},M1={},N1={}},
  {mm,nn}=Dimensions[L];
  A=FrmPoly[L];M1=FrmPoly[M];N1=FrmPoly[N];
  For[k=1,k<=nn-1,k++,
    L1=ZZP[A,M1,N1,k];Print["ZZP=",L1];
    L2=YYP[A,M1,N1,k];
    If[L1==L1*0, rez={L1,{1}}; rez=SimplP[L1,L2]];
    ZZP[A,M1,N1,k]=rez[[1]]; YYP[A,M1,N1,k]=rez[[2]];
    L2=Sum[rez[[2,j]](Variables[L][[1]])^(j-1),{j,1,Length[rez[[2]]]}];
    L1=Sum[rez[[1,j]](Variables[L][[1]])^(j-1),{j,1,Length[rez[[1]]]}];
    L1=VVP[A,M1,N1,k+1]; L2=WVP[A,M1,N1,k+1];
    rez=SimplP[L1,L2]; VVP[A,M1,N1,k+1]=rez[[1]];
    WVP[A,M1,N1,k+1]=rez[[2]];
  ];
  L1=ZZP[A,M1,N1,nn]; L2=YYP[A,M1,N1,nn]; rez=SimplP[L1,L2];
  Print["ZZP[",nn,"]=",rez[[1]]];
  Print["YYP[",nn,"]=",rez[[2]]];
  L2=Sum[rez[[2,j]](Variables[L][[1]])^(j-1),{j,1,Length[rez[[2]]]}];
  L1=Sum[rez[[1,j]](Variables[L][[1]])^(j-1),{j,1,Length[rez[[1]]]}];
  Return[Simplify[L1/L2//MatrixForm]];

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PolLCM[L_List]:= (* Find the least common multiple *)
Module[{m=m1=1,j},
  For[j=1,j<=Length[L],j++,
    If[Variables[L]!={},
      m=PolynomialLCM[m,L[[j]]],
      m=LCM[m,L[[j]]]
    ] ];
  If[Variables[m]=={},
    If[Length[m]!=0,
      For[j=1,j<=Length[m],j++, m1=LCM[m1,m[[j]]]
    ] ],
    If[Not[Head[m]!=List],
      For[j=1,j<=Length[m],j++, m1=PolynomialLCM[m1,m[[j]]] ],
      m1=m
    ] ];
  Return[Expand[m1] ];

SimplP[M1_List,M2_List]:=
Module[{p,q,r,vr={},M3=M4={},i},
  p=Sum[M1[[i+1]]*w^i,{i,0,Length[M1]-1}];
  q=Sum[M2[[i+1]]*w^i,{i,0,Length[M2]-1}];
  If[Head[q]!=List,r=Simplify[p/q],r=Simplify[p/q[[1]]];
  M3=PolLCM[Denominator[r]];
  If[Variables[M3]!={}, M4=Expand[Simplify[r*M3]];
  M3=Transpose[FrmPoly[{M3}]] [[1]];
  M4=FrmPoly[M4], M4=FrmPoly[r]; M3={1} ;
  Return[{M4,M3}];

Delt[L_List,M_List,N_List,i_]:=Delt[L,M,N,i]= (* Compute \Delta_{i,j+1} (3.15),(3.16) *)
Module[{L1={},L2={},L3={},L4={},L5={},L6={},L7={},gr,gr0,gr1,gr2,
  rez,del1,del2,del21,del22,del23,del24,del25},
  L1=YYP[L,M,N,i-1]; L2=InvNKP[N,i]; gr1=L2[[4]]; gr2=L2[[2]];
  L2=L2[[3]]; L3=NKP[N,i] [[1]]; L4=NKP[N,i] [[2]];
  L5=NKP[N,i] [[3]]; L6=DDP[L,M,N,i]; L7=FIP[L,M,N,i];
  If[L6=={}, L6={Transpose[L5[[1]]]*0}]; If[L7=={}, L7={L6[[1]]*0}];
  gr=2*Length[L1]+gr1-1; L1=DopZero[L1,gr]; L2=DopZero[L2,gr]; del1={};
  For[j=0,j<gr,j++, del1=Join[del1,{Sum[Sum[L1[[j-k-r+1]]L1[[k+1]]L2[[r+1]],
    {k,0,j-r}],{r,0,j}]]];

  gr0=3*Length[L1]+gr2+gr; L1=DopZero[L1,gr0];
  L2=DopZero[L2,gr0]; L3=DopZero[L3,gr0]; L4=DopZero[L4,gr0];
  L5=DopZero[L5,gr0]; L6=DopZero[L6,gr0]; L7=DopZero[L7,gr0];
  del2={}; del21={}; del22={}; del23={}; del24={}; del25={};
  If[i==2,For[j=0,j<gr0,j++,
    del21=Join[del21,{Sum[Sum[Sum[L4[[j-k-r-t+1]]L1[[k+1]]L1[[r+1]]L2[[t+1]],
      {k,0,j-t-r}],{r,0,j-t}],{t,0,j}]]];
    del22=Join[del22,Sum[Sum[Sum[Transpose[{L6[[j-k-r-t+1]]}].
      L3[[k+1]] [[1]]L6[[r+1]]L2[[t+1]],{k,0,j-t-r}],{r,0,j-t}],{t,0,j}]]];
    del23=Join[del23,Sum[Sum[Sum[Transpose[{L6[[j-k-r-t+1]]}].
      L5[[k+1]] [[1]]L1[[r+1]]L2[[t+1]],{k,0,j-t-r}],{r,0,j-t}],{t,0,j}]]];
    del24=Join[del24,Sum[Sum[Sum[Transpose[L5[[j-k-r-t+1]]].
      L6[[k+1]]L1[[r+1]]L2[[t+1]],{k,0,j-t-r}],{r,0,j-t}],{t,0,j}]]];
    del25=Join[del25,Sum[Sum[Transpose[L5[[j-k-r+1]]].
      L7[[k+1]]L1[[r+1]],{k,0,j-r}],{r,0,j}]]];
    For[j=0,j<gr0,j++,
      del21=Join[del21,{Sum[Sum[Sum[L4[[j-k-r-t+1]]L1[[k+1]]L1[[r+1]]L2[[t+1]],
        {k,0,j-t-r}],{r,0,j-t}],{t,0,j}]]];
      del22=Join[del22,Sum[Sum[Sum[Transpose[L6[[j-k-r-t+1]]].L3[[k+1]].
        L6[[r+1]]L2[[t+1]],{k,0,j-t-r}],{r,0,j-t}],{t,0,j}]]];
      del23=Join[del23,Sum[Sum[Sum[Transpose[L6[[j-k-r-t+1]]].
        L5[[k+1]]L1[[r+1]]L2[[t+1]],{k,0,j-t-r}],{r,0,j-t}],{t,0,j}]]];
      del24=Join[del24,Sum[Sum[Sum[Transpose[L5[[j-k-r-t+1]]].
        L6[[k+1]]L1[[r+1]]L2[[t+1]],{k,0,j-t-r}],{r,0,j-t}],{t,0,j}]]];
      del25=Join[del25,Sum[Sum[Transpose[L5[[j-k-r+1]]].
        L7[[k+1]]L1[[r+1]],{k,0,j-r}],{r,0,j}]]];
      del2=del21+del22+del23+del24+del25; del1=LastZeroP[del1]; del2=LastZeroP[del2];
      rez=SimplP[del1,del2];
    Return[rez];
  ] ];

```

```
(* ----- Compute inverse of N -----*)

NNinv[N_List,i_]:=NNinv[N,i]= (* Compute (3.33), (3.34) *)
Module[{L0={},L1={},L2={},L3={},L4={},L5={},Y={},e={},f={},g={},rez},
  L1=NKP[N,i]; L2=L1[[2]];
  If[i==1, Print["Ninv[" ,i,"]=",{{1}},L2]];Return[{{1}},L2];
  Y={}; L3=EEI[N,i][[2]]; L4=FFI[N,i][[2]]; L5=GII[N,i][[2]];
  L6=EEI[N,i][[1]]; L7=FFI[N,i][[1]]; L8=GII[N,i][[1]];
  gr=Length[L3]+Length[L4]+Length[L5];
  L3=DopZero[L3,gr]; L4=DopZero[L4,gr]; L5=DopZero[L5,gr];
  For[j=0,j<gr,j++,
    Y=Join[Y,Sum[Sum[L3[[j-k-r+1]]L4[[k+1,1]]L5[[r+1,1]],{k,0,j-r}],{r,0,j}]];
  Y=LastZeroP[Y]; e={};
  gr=Length[L6]+Length[L4]+Length[L5]; L6=DopZero[L6,gr];
  L4=DopZero[L4,gr]; L5=DopZero[L5,gr];
  For[j=0,j<gr,j++,
    e=Join[e,{Sum[Sum[L6[[j-k-r+1]]L4[[k+1,1]]L5[[r+1,1]],{k,0,j-r}],{r,0,j}]}];
  If[LastZeroP[e]==={},e={e[[1]]},e=LastZeroP[e]];
  f={}; gr=Length[L7]+Length[L3]+Length[L5]; L7=DopZero[L7,gr];
  L3=DopZero[L3,gr]; L5=DopZero[L5,gr];
  For[j=0,j<gr,j++,
    f=Join[f,{Sum[Sum[L7[[j-k-r+1]]L3[[k+1]]L5[[r+1,1]],{k,0,j-r}],{r,0,j}]}];
  If[LastZeroP[f]==={},f={f[[1]]},f=LastZeroP[f]];
  f=DopZero[f,Length[e]]; g={}; gr=Length[L8]+Length[L3]+Length[L4];
  L8=DopZero[L8,gr]; L3=DopZero[L3,gr]; L5=DopZero[L5,gr];
  For[j=0,j<gr,j++,
    g=Join[g,{Sum[Sum[L8[[j-k-r+1]]L3[[k+1]]L5[[r+1,1]],{k,0,j-r}],{r,0,j}]}];
  If[LastZeroP[g]==={},g={g[[1]]},g=LastZeroP[g]];
  iz=Frmpoly[FormE[e,f,g]]; rez=SimplP[iz,Y];
  Return[rez]]];

GII[N_List,i_]:=GII[N,i]= (* Compute (3.27), (3.28) *)
Module[{L0={},L1={},L2={},L3={},L4={},L5={},iz,iz1,iz2,gr,j,k,r,rez},
  L0=NNinv[N,i-1]; L1=NKP[N,i]; L2=L1[[2]]; L3=L1[[3]];
  L4=L0[[1]]; L5=L0[[2]]; gr=Length[L4]+Length[L5]+2*Length[N];
  L2=DopZero[L2,gr]; L4=DopZero[L4,gr]; L5=DopZero[L5,gr];
  L3=DopZero[L3,gr]; iz1={}; iz2={};
  For[j=0,j<gr,j++,
    iz2=Join[iz2,{Sum[Sum[L2[[j-k+1]]L5[[k+1]],{k,0,j}]}]];
  For[j=0,j<gr,j++,
    If[i<=2,iz1=Join[iz1,{Sum[Sum[-(Transpose[L3[[j-k-r+1]]].L4[[k+1]])
      .L3[[r+1]],{k,0,j-r}],{r,0,j}]}],
      iz1=Join[iz1,Sum[Sum[-Transpose[L3[[j-k-r+1]]].L4[[k+1]]
      .L3[[r+1]],{k,0,j-r}],{r,0,j}]}];
  iz=iz2+iz1;
  Return[{LastZeroP[L5],LastZeroP[iz]}]];

FFI[N_List,i_]:=FFI[N,i]= (* Compute (3.29), (3.30) *)
Module[{L0={},L1={},L2={},L3={},L4={},L5={},iz,iz1,gr,j,k,r,rez},
  L0=NNinv[N,i-1]; L1=NKP[N,i]; L2=GII[N,i][[2]];
  L3=L1[[3]]; L4=L0[[1]]; L5=L0[[2]]; gr=Length[L4]+Length[N];
  iz={}; L3=DopZero[L3,gr]; L4=DopZero[L4,gr];
  For[j=0,j<gr,j++, iz=Join[iz,{Sum[L4[[j-k+1]].L3[[k+1]],{k,0,j}]}];
  If[LastZeroP[iz]==={},iz={iz[[1]]},iz=LastZeroP[iz]];
  Return[{-iz,LastZeroP[L2]}]];

EEI[N_List,i_]:=EEI[N,i]= (* Compute (3.31), (3.32) *)
Module[{L0={},L1={},L2={},L3={},L4={},L5={},L6={},L7={},L8={},
  s1={},s2={},iz,iz1,gr,j,k,r,rez},
  L0=NNinv[N,i-1]; L1=L0[[1]]; L2=L0[[2]];
  L5=FFI[N,i][[1]]; L6=FFI[N,i][[2]]; L7=GII[N,i][[1]]; L8=GII[N,i][[2]];
  gr=Max[Length[L1]+Length[L7]+2*Length[L8],Length[L2]+Length[L8]+2*Length[L7]];
  iz={}; L1=DopZero[L1,gr]; L7=DopZero[L7,gr]; L6=DopZero[L6,gr];
  L5=DopZero[L5,gr]; L2=DopZero[L2,gr]; s1={};
  For[j=0,j<gr,j++,
    If[Length[Dimensions[L5[[1]]]]==1,
      s1=Join[s1,{Sum[Sum[L1[[j-k-r+1]]L7[[k+1]]L6[[r+1]],{k,0,j-r}],{r,0,j}]}];
      s1=Join[s1,{Sum[Sum[L1[[j-k-r+1]]L7[[k+1]]L6[[r+1,1]],{k,0,j-r}],{r,0,j}]}]]];
```

```

s2={};
For[j=0,j<gr,j++,
  If[Length[Dimensions[L5[[1]]]]==1,
    s2=Join[s2,{{Sum[Sum[L2[[j-k-r+1]]L5[[k+1]]L5[[r+1]],{k,0,j-r}],{r,0,j}]}},
    s2=Join[s2,{Sum[Sum[L2[[j-k-r+1]]L5[[k+1]].Transpose[L5[[r+1]]],{k,0,j-r}],{r,0,j}]}]]];
iz=s1+s2; gr1=Length[L7]+Length[L2]+Length[L6]; iz1={};
L2=DopZero[L2,gr1]; L7=DopZero[L7,gr1]; L6=DopZero[L6,gr1];
For[j=0,j<gr1,j++,
  iz1=Join[iz1,{Sum[Sum[L2[[j-k-r+1]]L7[[k+1]]L6[[r+1,1]],{k,0,j-r}],{r,0,j}]}]]];
Return[{LastZeroP[iz],LastZeroP[iz1]}]];

FormE[e_List,f_List,g_List]:=
Module[{e1,f1,g1,Y,Y1,Y2,i},
e1=Sum[e[[i+1]]*w^i,{i,0,Length[e]-1}];
f1=Sum[f[[i+1]]*w^i,{i,0,Length[f]-1}];
g1=Sum[g[[i+1]]*w^i,{i,0,Length[g]-1}];
If[Head[g1]!=List,g1={g1}];
Y1=0*e1;
For[j=1,j<=Length[e1],j++,
  If[Length[f1]==1,Y1[[j]]=Append[e1[[j]],f1[[j]]],
  Y1[[j]]=Join[e1[[j]],f1[[j]]]];
If[Length[f1]==1,Y2={Join[f1,g1]},
  Y2={Join[Transpose[f1][[1]],g1]}];
Y=Join[Y1,Y2];
Return[Y];

InvNKP[L_List,i_]:=InvNKP[L,i]=Module[{L0={},rez,iz}, (* Compute (3.35) *)
L0=NNinv[L,i-1]; rez=SimplP[L0[[1]],L0[[2]]];
If[i==2,iz={rez[[1]]},Length[rez[[1]]]-1,Transpose[{rez[[2]]},Length[rez[[2]]]-1],
  iz={rez[[1]],Length[rez[[1]]]-1,Transpose[{rez[[2]]},Length[rez[[2]]]-1]}];
Return[iz]];

```

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